Decomposition rules for conformal pairs associated to symmetric spaces and abelian subalgebras of $\mathbb{Z}_2$-graded Lie algebras

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Abstract

We give uniform formulas for the branching rules of level 1 modules over orthogonal affine Lie algebras for all conformal pairs associated to symmetric spaces. We also provide a combinatorial intepretation of these formulas in terms of certain abelian subalgebras of simple Lie algebras.

1 Introduction

A pair $(\mathfrak{s}, \mathfrak{k})$, where $\mathfrak{s}$ is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{s}$, such that the restriction of the Killing form of $\mathfrak{s}$ to $\mathfrak{k}$ is non-degenerate, is called a *conformal pair* if there exists an integrable highest weight module V over the affine Kac–Moody algebra $\widehat{\mathfrak{s}}$, faithful on each simple component of $\mathfrak{s}$, such that the restriction to $\widehat{\mathfrak{k}}$ of each weight space of the center of $\mathfrak{k}$ in V decomposes into a finite direct sum of irreducible $\widehat{\mathfrak{k}}$-modules. In such a case $\mathfrak{k}$ is called a *conformal subalgebra* of $\mathfrak{s}$.

It is well-known that any integrable highest weight $\widehat{\mathfrak{s}}$-module, when restricted to $\widehat{\mathfrak{k}}$, decomposes into a direct sum of irreducible $\widehat{\mathfrak{k}}$-modules [12], but almost always this decomposition is infinite.

The first cases of a finite decomposition, found in [13], are as follows. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the eigenspace decomposition of an inner involution of a simple Lie algebra $\mathfrak{g}$ such that $\mathfrak{k}$ is semisimple. This defines an embedding $\mathfrak{k} \subset so(\mathfrak{p})$. It was shown by Kac and Peterson in [13], by an explicit decomposition formula, that the restriction of the spinor representations of $\widehat{so(\mathfrak{p})}$ to $\widehat{\mathfrak{k}}$ is a finite direct sum of irreducible $\widehat{\mathfrak{k}}$-modules. Thus, the pair $(so(\mathfrak{p}), \mathfrak{k})$ is conformal.

Due to their importance for string compactifications, a series of papers on conformal pairs appeared in the second half of the 1980s in physics literature. First of all, a connection to representation theory of the Virasoro algebra was established. Namely, it was found that the decomposition in question is finite if and only if the following numerical criterion holds: the central charges of the Sugawara construction of the Virasoro algebra for $\widehat{\mathfrak{s}}$ and $\widehat{\mathfrak{k}}$ are equal [9]. This immediately has led to a conclusion: the decomposition in question has a chance to be finite only if the level of the $\widehat{\mathfrak{s}}$-module V is equal to 1, and if it is finite for one of the $\widehat{\mathfrak{s}}$-modules of level 1, it is also finite for all others. Furthermore, Goddard, Nahm and Olive show [9] that the observation of Kac and Peterson can be reversed. Namely all conformal pairs $(so(\mathfrak{p}), \mathfrak{k})$ are obtained from an involution (not necessarily inner) of a semisimple Lie algebra $\mathfrak{g}$, and all such pairs are conformal. However, they obtain this result using the above numerical criterion, and do not find actual decompositions.

All conformal subalgebras $\mathfrak{k}$ for all simple Lie algebras $\mathfrak{s}$ were classified in [3] and [24] by making use of the numerical criterion. Also, it was pointed out in [2] that, using the conformal pairs $(so_{2n}, g\ell_n)$ and $(so_{4n}, sp_{2n} \times s\ell_2)$, one can reduce the study of conformal subalgebras in all classical Lie algebras to that in so_n.

Around the same time the general problem of restricting representations of affine Lie algebras to their subalgebras was treated, using the theory of modular forms. Namely it was observed in [14] that the branching rules are described by certain modular functions, called branching functions, for which one can write down explicit transformation formulas. This idea was further developed in [16], where the above mentioned "modular constraints", along with the "conformal constraints", provided by the Virasoro algebra, allowed to compute easily branching functions (which are constants in the conformal pair case) in many interesting cases, and, in principle, in any given case. The technology, developed in [16] was subsequently used in [15] to find all the decompositions of all integrable highest weight modules of level 1 over affine Lie algebras $\widehat{\mathfrak{s}}$, restricted to affine subalgebras $\widehat{\mathfrak{k}}$, where $\mathfrak{k}$ is a conformal subalgebra of a simple exceptional Lie algebra $\mathfrak{s}$. The branching rules of some other conformal embeddings were subsequently found in [1], [20] and a few other papers, written in the 1990s.

The problem of finding a general conceptual formula for branching rules for level 1 integrable highest weight modules over $\widehat{so(\mathfrak{p})}$, when restricted to $\widehat{\mathfrak{k}}$, where the conformal embedding of $\mathfrak{k}$ in $so(\mathfrak{p})$ is defined by the eigenspace decomposition of an involution of a semisimple Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, has remained an open problem.

In the present paper we completely solve this problem. The solution

turned out to be intimately related to recent developments in the study of abelian subalgebras of simple Lie algebras, that began with a paper of Kostant [17] and continued in [4], [5], [6], [7], [18], [21], [22], [25].

Let us explain our main observation on the example of a conformal embedding $\mathfrak{k} \subset so(\mathfrak{k})$, where $\mathfrak{k}$ is a simple Lie algebra, via the adjoint representation of $\mathfrak{k}$. In this case the restriction to $\widehat{\mathfrak{k}}$ of the basic + vector representations of $\widehat{so(\mathfrak{k})}$ decomposes into a direct sum of $2^{\text{rank}\,\mathfrak{k}}$ irreducible $\widehat{\mathfrak{k}}$-modules (this decomposition was found already by Kac and Wakimoto [16]), and, remarkably, these $2^{\text{rank}\,\mathfrak{k}}$ $\widehat{\mathfrak{k}}$-modules are in a canonical one-to-one correspondence with all abelian ideals of a Borel subalgebra of $\mathfrak{k}$.

Our main result is that for all conformal pairs associated to symmetric spaces, the decomposition of level 1 modules over $\widehat{so(\mathfrak{p})}$ is described in terms of a certain class of abelian subalgebras of semisimple $\mathbb{Z}_2$-graded Lie algebras $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, studied and classified recently by Cellini, Möseneder Frajria and Papi [4].

We hope that the connection of representation theory of affine Lie algebras to the theory of abelian subalgebras of simple Lie algebras will shed a new light on the latter theory as well. So far we obtained only partial results in this direction.

Now we describe our results in a special case which might give the flavour of the general case. First remark that $\widehat{so(\mathfrak{p})}$ is an affine algebra of type $B^{(1)}$ or $D^{(1)}$ according to whether $\dim(\mathfrak{p})$ is odd or even. Hence the level 1 modules are the fundamental representations associated to the extremal nodes of the Dynkin diagram. They are the basic, vector and spin representations, and have been studied since a long time.

We consider in detail the case of the basic and vector representation of $\widehat{\mathfrak{k}}$ and furthermore we assume that $\mathfrak{k}$ is semisimple. Denote by $L(\tilde{\Lambda}_0)$ the basic representation, by $L(\tilde{\Lambda}_1)$ the vector representation and set $L = L(\tilde{\Lambda}_0)+L(\tilde{\Lambda}_1)$. The first step in our analysis consists in calculating the character of the $\widehat{\mathfrak{k}}$-module L. This is done using the explicit description of the action given in [13]. To be more precise we need to fix some notation. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{k}$; denote by $\Delta_{\mathfrak{k}}$ the set of $\mathfrak{h}_0$-roots of $\mathfrak{k}$ and by $\Delta(\mathfrak{p})$ the set of $\mathfrak{h}_0$-weights of $\mathfrak{p}$. Fix a set of positive roots $\Delta_{\mathfrak{k}}^+$ and let $\mathfrak{b}_0$ be the corresponding Borel subalgebra. Let σ denote the involution which induces the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and denote by $\hat{L}(\mathfrak{g}, \sigma)$ the extended loop algebra associated to the pair $(\mathfrak{g}, \sigma)$ (see [12], Ch. 8) and by $\widehat{W}$ its Weyl group. The choice of $\Delta_{\mathfrak{k}}^+$ induces natural choices $\widehat{\Delta}^+$, $\widehat{\Delta}_{\mathfrak{k}}^+$ for the positive roots of $\hat{L}(\mathfrak{g}, \sigma)$, $\widehat{\mathfrak{k}}$ respectively. Here, as above, $\widehat{\mathfrak{k}}$ denotes the untwisted affine algebra associated to $\mathfrak{k}$; we also set $\delta_{\mathfrak{k}}$ to be its fundamental imaginary root and $\widehat{W}_{\mathfrak{k}}$

its Weyl group. Finally let Δ^+ denote a set of positive roots (w.r.t. the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$) of $\mathfrak{g}$ compatible with that of $\mathfrak{k}$ (see 2.3). This allows us to define

$$\begin{aligned}
\Delta^+(\mathfrak{p}) &= \Delta^+_{|\mathfrak{h}_0} \cap \Delta(\mathfrak{p}),\\
\widehat{\Delta}^+(\mathfrak{p}) &= \{(m+\frac{1}{2})\delta_{\mathfrak{k}} + \alpha \mid \alpha \in \Delta^+(\mathfrak{p}),\, m \in \mathbb{Z}\}.
\end{aligned}$$

Then it turns out that, up to an exponential factor, $ch(L)$ equals

$$\prod_{\alpha \in \widehat{\Delta}^+(\mathfrak{p})} \left(1+e^{-\alpha}\right)^{mult(\alpha)}.$$

(see (3.4)). To extract from the previous formula information on the $\widehat{\mathfrak{k}}$-module structure of L we generalize an idea used in [23] in the finite dimensional equal rank case. We introduce a natural map $\psi_0 : \widehat{\mathfrak{k}} \to \widehat{L}(\mathfrak{g},\sigma)$, whose transpose induces a bijection $\psi_0^* : \widehat{\Delta} \leftrightarrow \widehat{\Delta}^+_{\mathfrak{k}} \cup \widehat{\Delta}^+(\mathfrak{p})$. Using this map and the Weyl-Kac character formula we get the following decomposition into irreducible $\widehat{\mathfrak{k}}$-modules

$$L(\tilde{\Lambda}_\epsilon) = \sum_{\substack{u \in W'_{\sigma,0}\\ \ell(u) \equiv \epsilon \, mod 2}} L(\psi_0^*(u\widehat{\rho}) - \widehat{\rho}_{\mathfrak{k}} + \frac{1}{2}\epsilon\delta_{\mathfrak{k}}). \tag{1.1}$$

Here $W'_{\sigma,0}$ is the set of minimal right coset representatives of $(\psi_0^*)^{-1}\widehat{W}_{\mathfrak{k}}\psi_0^*$ in $\widehat{W}$, $\widehat{\rho}, \widehat{\rho}_{\mathfrak{k}}$ are the sum of fundamental weights in $\widehat{\Delta}^+, \widehat{\Delta}^+_{\mathfrak{k}}$ and $\epsilon = 0, 1$. A more accurate statement of formula (1.1) is given in Theorem 3.5.

The combinatorial interpretation of formula (1.1) arises from the fact that, if C_1 denotes the fundamental alcove of $\widehat{W}$, then the set $\bigcup_{w \in W'_{\sigma,0}} w\overline{C_1}$ is the polytope studied in [4] in connection with the problem of enumerating $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{p}$. This coincidence allows us to give a much more explicit rendering of formula (1.1). For instance, if σ is an automorphism of type $(0,\ldots,1,\ldots,0;1)$ with 1 in a position corresponding to a short root, then we have

$$L(\tilde{\Lambda}_\epsilon) = \bigoplus_{\substack{A \in \Sigma\\ |A| \equiv \epsilon \ mod\, 2}} L\left(\Lambda_{0,\mathfrak{k}} + \langle A\rangle - \frac{1}{2}(|A| - \epsilon)\delta_{\mathfrak{k}}\right),$$

where Σ is the set of $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{p}$, and for $A \in \Sigma$, $\langle A\rangle$, $|A|$ denote the sum (resp. the number) of the roots in A. The general case is completely described in Theorem 3.9.

The situation in the case of the spin representation ($\mathfrak{k}$ semisimple) is much more complex. For instance, we need to use the factorization of the involution σ as $\mu\eta$ with η inner and μ diagram automorphism to write down the auxiliary map $\psi_1 : \widehat{\mathfrak{k}} \to L'(\mathfrak{g}, \sigma)$ which plays the role of ψ_0 in the spin case and which is the crucial tool for manipulating the character. Moreover the target algebra $L'(\mathfrak{g}, \sigma)$ is different according to whether we are considering the equal rank case, the $A_{2n}^{(2)}$ case or the remaining non equal rank cases. Surprisingly enough, we obtain a decomposition formula which is quite similar to (1.1):

$$ch(X_r) = 2^{\lfloor \frac{N-n}{2} \rfloor} \sum_{u \in W'_{\sigma,1}} ch(L(a_0\psi_1^*(u\widehat{\rho}') - \widehat{\rho}_{\mathfrak{k}})). \tag{1.2}$$

We refer the reader to Proposition 4.7 for the undefined notation. As far as the combinatorial interpretation is concerned, we get again a description of formula (1.2) in terms of abelian subalgebras, but in the non equal rank the right class to consider is that of *noncompact* subalgebras (cf. Definition 4.3) which are stable under the Borel subalgebra $\mathfrak{b}_0 \cap \mathfrak{k} \cap \mathfrak{k}_\mu$ of the subalgebra $\mathfrak{k} \cap \mathfrak{k}_\mu$ of μ-fixed points in $\mathfrak{k}$. The relevant results in this direction are Theorems 4.10, 4.12, 4.13.

A few words on the case in which $\mathfrak{k}$ has a non-trivial center. In this case we describe the finite decomposition of an eigenspace of the center on the level 1 modules. Also in this case there is a special subset of abelian stable subalgebras of $\mathfrak{p}$ which plays an important role in the description of the decompositions. We describe in detail the finite decomposition of each eigenspace of the center (see Theorems 5.4, 5.5).

The paper is organized as follows. In Section 2 we recall the necessary information on the structure and representation theory of affine Lie algebras, as well as the construction of all level 1 modules over the affinization of orthogonal Lie algebras so_n. In Section 3 we find the decompositions of the basic + vector representations of $\widehat{so(\mathfrak{p})}$, restricted to $\widehat{\mathfrak{k}}$, $\mathfrak{k}$ semisimple, and in Section 4 we solve the same problem for the spinor representations. In Section 5 we deal with the case when $\mathfrak{k}$ has a non-trivial center. Finally in Section 6 we consider some concrete examples and discuss connections of the theory of abelian subalgebras to modular invariance.

2 Preliminaries

2.1 Lie algebra involutions and affine algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, σ an involutive automorphism of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding eigenspace decomposition. Let $\mathfrak{h}_0$ be a Cartan subalgebra of the reductive subalgebra $\mathfrak{k}$ and let $\mathfrak{z}$ be the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$. Let $(\cdot,\cdot)$ denote a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Then (see [10], Lemma 5.3 or [12] Lemma 8.1), $\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}$ and $(\mathfrak{k},\mathfrak{p}) = 0$. In particular $(\cdot,\cdot)_{|\mathfrak{k}\times\mathfrak{k}}$ is a nondegenerate invariant form on $\mathfrak{k}$ and $(\cdot,\cdot)$ is nondegenerate when restricted to $\mathfrak{h}_0$, so we can induce a form, still denoted by $(\cdot,\cdot)$ on $\mathfrak{h}_0^*$.

Consider the root system $\Delta_{\mathfrak{k}}$ of the pair $(\mathfrak{k},\mathfrak{h}_0)$ and fix a subset of positive roots $\Delta_{\mathfrak{k}}^+$. Let $\mathfrak{b}_0$ denote the corresponding Borel subalgebra of $\mathfrak{k}$. We denote by $\Delta(\mathfrak{p})$ the set of $\mathfrak{h}_0$-weights of $\mathfrak{p}$.

Let $L(\mathfrak{g})$ be the loop algebra of $\mathfrak{g}$:

$$L(\mathfrak{g}) = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}.$$

Let $L(\mathfrak{g},\sigma)$ be the subalgebra

$$L(\mathfrak{g},\sigma) = \left(\sum_{n\in\mathbb{Z},\ n \text{ even}} t^n \otimes \mathfrak{k}\right) \oplus \left(\sum_{n\in\mathbb{Z},\ n \text{ odd}} t^n \otimes \mathfrak{p}\right)$$

and consider the extended loop algebra $\widehat{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}K' \oplus \mathbb{C}d'$ with bracket defined by

$$\begin{aligned}&[t^n \otimes X + \lambda K' + \mu d', t^m \otimes Y + \lambda_1 K' + \mu_1 d'] = \\ &\quad = t^{n+m} \otimes [X,Y] + \delta_{n,-m} n (X,Y) K' + \mu_1 n t^n \otimes X + \mu m t^m \otimes Y. \end{aligned} \tag{2.1}$$

(The construction of the extended loop algebra is done in [12] only for $\mathfrak{g}$ simple, but everything extends to semisimple $\mathfrak{g}$ in a straightforward way). Set $\widehat{L}(\mathfrak{g},\sigma) = L(\mathfrak{g},\sigma) \oplus \mathbb{C}K' \oplus \mathbb{C}d'$. Clearly $\widehat{L}(\mathfrak{g},\sigma)$ is a subalgebra of $\widehat{L}(\mathfrak{g})$.

Set $\widehat{\mathfrak{h}} = (1 \otimes \mathfrak{h}_0) \oplus \mathbb{C}K' \oplus \mathbb{C}d'$ and let $\widehat{\Delta}$ denote the set of nonzero $\widehat{\mathfrak{h}}$-weights of $\widehat{L}(\mathfrak{g},\sigma)$. Define $\delta' \in \widehat{\mathfrak{h}}^*$ by setting

$$\delta'(1 \otimes \mathfrak{h}_0) = \delta'(K') = 0 \quad \delta'(d') = 1.$$

We identify $\mathfrak{h}_0^*$ and the subset $\{\lambda \in \widehat{\mathfrak{h}}^* \mid \lambda(d') = \lambda(K') = 0\}$ of $\widehat{\mathfrak{h}}^*$.

Notation. If $\lambda \in \widehat{\mathfrak{h}}^*$ we denote by $\overline{\lambda}$ its restriction to $1 \otimes \mathfrak{h}_0$.

The set of roots of $\widehat{L}(\mathfrak{g},\sigma)$ is

$$\begin{aligned}\widehat{\Delta} =&\{k\delta'+\alpha \mid \alpha\in\Delta_{\mathfrak{k}},\ k \text{ even}\}\cup\{k\delta'+\alpha \mid \alpha\in\Delta(\mathfrak{p}),\ k \text{ odd}\}\\ &\cup\{k\delta' \mid k\in 2\mathbb{Z},\ k\neq 0\}.\end{aligned}$$

We set $\widehat{\Delta}^+=\Delta^+_{\mathfrak{k}}\cup\{\alpha\in\widehat{\Delta}\mid \alpha(d')>0\}$. Let $\widehat{\Pi}$ be the corresponding set of simple roots. Denote by $\widehat{W}$ the Weyl group generated by $\widehat{\Pi}$.

Assume now that σ is indecomposable (i.e. $\mathfrak{g}$ has no non-trivial σ-stable ideals). Then either $\mathfrak{g}$ is simple, or $\mathfrak{g}$ is a direct sum of two copies of a simple Lie algebra $\mathfrak{k}$ and σ permutes the summands. We call the latter the *complex* case.

The following two propositions give a summary of Exercises 8.1–8.4 from [12]. The proof can be found in § 5 of [10], Ch.X.

Proposition 2.1. *1. $|\widehat{\Pi}|=n+1$, where n is the rank of $\mathfrak{k}$.*

2. If $\widehat{\Pi}=\{\alpha_0,\dots,\alpha_n\}$, then $\overline{\alpha}_0,\dots,\overline{\alpha}_n$ span $\mathfrak{h}_0$.

3. $(\overline{\alpha}_i,\overline{\alpha}_i)>0$ for all i and

$$a_{ij}=2\frac{(\overline{\alpha}_i,\overline{\alpha}_j)}{(\overline{\alpha}_i,\overline{\alpha}_i)}\in-\mathbb{Z}_+,\ \text{if } i\neq j.$$

4. $A=(a_{ij})$ is a generalized Cartan matrix of an affine type.

We label the α_i so that the corresponding Dynkin diagram is one of those displayed at pp. 54–55 of [12].

If $\alpha\in\widehat{\Delta}$, then α can be written uniquely as $\sum_{i=0}^n m_i(\alpha)\alpha_i$ with $m_i(\alpha)\in\mathbb{Z}$. Write $\alpha_i=s_i\delta'+\overline{\alpha}_i$. By Proposition 2.1.3, $\overline{\alpha}_i\neq 0$. Let $h_{\overline{\alpha}_i}$ be the unique element of $\mathfrak{h}_0$ such that $\overline{\alpha}_i(h)=(h_{\overline{\alpha}_i},h)$ for all $h\in\mathfrak{h}_0$. Set $h_i=\frac{2}{(\overline{\alpha}_i,\overline{\alpha}_i)}h_{\overline{\alpha}_i}$ and fix $t^{s_i}\otimes X_i\in\widehat{L}(\mathfrak{g},\sigma)_{\alpha_i}$, $t^{-s_i}\otimes Y_i\in\widehat{L}(\mathfrak{g},\sigma)_{-\alpha_i}$ in such a way that $(X_i,Y_i)=\frac{2}{(\overline{\alpha}_i,\overline{\alpha}_i)}$. Then $[X_i,Y_i]=h_i$. It follows that

$$[t^{s_i}\otimes X_i,t^{-s_i}\otimes Y_i]=\frac{2s_i}{(\overline{\alpha}_i,\overline{\alpha}_i)}K'+h_i.$$

Set $\alpha_i^\vee=\frac{2s_i}{(\overline{\alpha}_i,\overline{\alpha}_i)}K'+h_i$ and $\widehat{\Pi}^\vee=\{\alpha_0^\vee,\dots,\alpha_n^\vee\}$. In the following proposition we use the notation of [12], Ch. 1.

Proposition 2.2. *The triple $(\widehat{\mathfrak{h}},\widehat{\Pi},\widehat{\Pi}^\vee)$ is a realization of A and the map*

$$e_i\mapsto t^{s_i}\otimes X_i\quad f_i\mapsto t^{-s_i}\otimes Y_i$$

extends to a Lie algebra isomorphism of the affine Kac-Moody algebra $\mathfrak{g}(A)$ to the Lie algebra $\widehat{L}(\mathfrak{g},\sigma)$.

Let $a_0,\dots,a_n$ (resp. $a_0^\vee,\dots,a_n^\vee$) be positive integers with $G.C.D(a_0,\dots,a_n)=1$ that are coefficients of a linear dependence between the rows (resp. columns) of the matrix A: $\sum\limits_{i=0}^n a_i\overline{\alpha}_i=0$ and $\sum\limits_{i=0}^n a_i^\vee h_i=0$. Set $\delta=\sum_{i=0}^n a_i\alpha_i$ and notice that $\delta=(\sum_{i=0}^n a_is_i)\delta'$. We also let $K=\sum_{i=0}^n a_i^\vee\alpha_i^\vee$ be the canonical central element.

Proposition 2.3. *Set $k=\frac{2}{\sum_{i=0}^n a_is_i}$. Then $k=1$ if σ is inner and $k=2$ otherwise.*

Proof. Since $2\delta'=k\delta$ is a root (cf. [12], Theorem 5.6 b)) we deduce that k is an integer, hence $k\in\{1,2\}$. Since $\delta'=\frac{k}{2}\delta$ is a root if and only if $\mathfrak{z}\neq\mathfrak{h}_0$, we see that $k=2$ if and only if σ is not of inner type. □

Remark 2.1. If $\mathfrak{g}$ is simple of type X_N, then $\widehat{L}(\mathfrak{g},\sigma)$ is an affine Kac-Moody algebra of type $X_N^{(k)}$. If $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{k}$, where $\mathfrak{k}$ is simple of type X_N, then $\widehat{L}(\mathfrak{g},\sigma)$ is of type $X_N^{(1)}$.

Remark 2.2. If $\mathfrak{g}$ is simple, using the terminology of [12], we have that σ is an automorphism of type $(s_0,\dots,s_n;k)$. In the complex case we have $k=2$, $s_0=1$.

If $\mathfrak{g}$ is simple, we choose $(\cdot,\cdot)=k^{-1}(\cdot,\cdot)_n$, where $(\cdot,\cdot)_n$ is the invariant form on $\mathfrak{g}$ such that the square root lenght of a long root is 2. We will call $(\cdot,\cdot)_n$ a *normalized* invariant form. If $\mathfrak{g}=\mathfrak{k}\times\mathfrak{k}$ we define $(\cdot,\cdot)$ by

$$((X,Y),(X',Y'))=\tfrac{1}{2}((X,X')_n+(Y,Y')_n),$$

where $(\cdot,\cdot)_n$ is the normalized invariant form on $\mathfrak{k}$.

We define a standard invariant form $(\cdot,\cdot)$ on $\widehat{L}(\mathfrak{g},\sigma)$ by setting

$$(\alpha_i^\vee,h)=\frac{2}{(\overline{\alpha}_i,\overline{\alpha}_i)}\alpha_i(h)\quad\text{for } i=0,\dots,n \text{ and } h\in\widehat{\mathfrak{h}} \tag{2.2}$$

$$(d',d')=0. \tag{2.3}$$

We want to prove that the previous formulas define a normalized invariant form on $\widehat{L}(\mathfrak{g},\sigma)$, i.e. $(\theta,\theta)=2$, where $\theta=\sum_{i=1}^n a_i\alpha_i$. Let $\nu:\widehat{\mathfrak{h}}\to\widehat{\mathfrak{h}}^*$ be the induced isomorphism and let $(\cdot,\cdot)$ be the induced form on $\widehat{\mathfrak{h}}^*$. Since

$$\alpha_i(h)=\frac{(\overline{\alpha}_i,\overline{\alpha}_i)}{2}(\alpha_i^\vee,h)$$

we see that $\nu^{-1}(\alpha_i)=\frac{(\overline{\alpha}_i,\overline{\alpha}_i)}{2}\alpha_i^\vee$. It follows that

$$(\alpha_i,\alpha_j)=(\overline{\alpha}_i,\overline{\alpha}_j),$$

hence, if $\lambda, \mu \in Span(\widehat{\Pi})$, then

$$(\lambda, \mu) = (\overline{\lambda}, \overline{\mu}). \tag{2.4}$$

We need to discuss the relationship between the roots of $\mathfrak{g}$ and the roots of $\widehat{L}(\mathfrak{g}, \sigma)$. Write $\mathfrak{z} = \mathfrak{h}_0 \oplus \mathfrak{a}_{\mathbb{C}}$, where $\mathfrak{a}_{\mathbb{C}} = \mathfrak{p} \cap \mathfrak{z}$. Let Δ be the $\mathfrak{z}$-root system of $\mathfrak{g}$. There are three types of roots in Δ: those such that $\alpha_{|\mathfrak{a}_{\mathbb{C}}} = 0$ and whose root vector X_α is in $\mathfrak{k}$, those such that $\alpha_{|\mathfrak{a}_{\mathbb{C}}} = 0$ and whose root vector X_α is in $\mathfrak{p}$, and those such that $\alpha_{|\mathfrak{a}_{\mathbb{C}}} \neq 0$. These are usually called respectively compact imaginary roots, noncompact imaginary (or singular imaginary) roots, and complex roots. To avoid confusion with standard Kac-Moody terminology we call them compact, noncompact, and complex. If α is a complex root, then the corresponding root vector decomposes as

$$X_\alpha = u_\alpha + v_\alpha$$

with $u_\alpha \in \mathfrak{k}$ and $v_\alpha \in \mathfrak{p}$. Then u_α is a root vector in $\mathfrak{k}$ for the root $\alpha_{|\mathfrak{h}_0}$ and v_α is a weight vector in $\mathfrak{p}$ for the weight $\alpha_{|\mathfrak{h}_0}$ in $\Delta(\mathfrak{p})$. In particular α is a complex root if and only if $\alpha_{|\mathfrak{h}_0} \in \Delta_{\mathfrak{k}} \cap \Delta(\mathfrak{p})$. It follows that $\alpha \in \Delta$ is a compact root if and only if $\alpha_{|\mathfrak{h}_0} \in \widehat{\Delta}$ and $\delta' + \alpha_{|\mathfrak{h}_0} \notin \widehat{\Delta}$, $\alpha \in \Delta$ is a noncompact root if and only if $\alpha_{|\mathfrak{h}_0} \notin \widehat{\Delta}$ and $\delta' + \alpha_{|\mathfrak{h}_0} \in \widehat{\Delta}$, and $\alpha \in \Delta$ is a complex root if and only if $\alpha_{|\mathfrak{h}_0} \in \widehat{\Delta}$ and $\delta' + \alpha_{|\mathfrak{h}_0} \in \widehat{\Delta}$. More precisely if $k = 1$, then $\mathfrak{h}_0 = \mathfrak{z}$ hence, if $\alpha \in \widehat{\Delta}$, then $\overline{\alpha} = \beta_{|\mathfrak{h}_0}$ with β compact or noncompact. It follows that $(\alpha, \alpha) = (\overline{\alpha}, \overline{\alpha}) = (\beta, \beta)_n$. In particular, since α_0 is a long root, $(\alpha_0, \alpha_0) = 2$. If $k = 2$ and $\mathfrak{g}$ is simple, then $\delta' = \delta$ hence, if $\alpha \in \widehat{\Delta}$, then $\overline{\alpha} = \beta_{|\mathfrak{h}_0}$ with β compact or noncompact if and only if α is a long root. It follows that if α is a long root and $\overline{\alpha} = \beta_{|\mathfrak{h}_0}$, then $(\alpha, \alpha) = (\overline{\alpha}, \overline{\alpha}) = k(\beta, \beta)_n = 4$. In particular, since α_0 is a short root, $a_0(\alpha_0, \alpha_0) = 2$. In the complex case one checks directly that $(\alpha_0, \alpha_0) = 2$ in this case too. We have proven

Lemma 2.4. *The form $(\cdot, \cdot)$ on $\widehat{L}(\mathfrak{g}, \sigma)$ defined above is a normalized standard invariant form.*

Since $(d', K) = \sum a_i^\vee \frac{2}{(\alpha_i, \alpha_i)} \alpha_i(d') = \sum a_i^\vee \frac{2 s_i}{(\alpha_i, \alpha_i)}$, and $K = \sum_{i=0}^n \frac{2 a_i^\vee s_i}{(\alpha_i, \alpha_i)} K'$ we see that $(d', K') = 1$. Also remark that

$$(d', h) = 0 \text{ if } h \in \mathfrak{h}_0 \tag{2.5}$$

which is easily proved by observing that

$$(d', h_i) = (d', \alpha_i^\vee - 2\frac{s_i}{(\alpha_i, \alpha_i)} K') = \frac{2}{(\alpha_i, \alpha_i)} \alpha_i(d') - 2\frac{s_i}{(\alpha_i, \alpha_i)} = 0.$$

Let H be the unique element of $\mathfrak{h}_0$ such that $\overline{\alpha_i}(H) = s_i$ for $i = 1, \dots, n$. Then easy calculations show that $d = \frac{a_0 k}{2}(d' - H - \frac{1}{2}(H,H)K')$ is a scaling element for $\widehat{L}(\mathfrak{g}, \sigma)$ and $(d,d) = 0$. It follows that, if we define $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ by setting $\Lambda_0(\alpha_i^\vee) = \delta_{i0}$ and $\Lambda_0(d) = 0$, then the form $(\cdot,\cdot)$ on $\widehat{L}(\mathfrak{g}, \sigma)$ is given by the formulas of [12, § 6.2]. In particular we see that $\sum 2\frac{s_i a_i^\vee}{(\alpha_i,\alpha_i)} = \sum a_i s_i = \frac{2}{k}$. Hence $K = \frac{2}{k}K'$.

2.2 The Lie algebra $\widehat{\mathfrak{k}}$ and the character formula

In general $\mathfrak{k}$ is a reductive Lie algebra, hence we can write $\mathfrak{k} = \mathfrak{k}_0 \oplus \sum_{S=1}^{M} \mathfrak{k}_S$, where $\mathfrak{k}_0$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_S$ are the simple ideals of $\mathfrak{k}$. If $S > 0$, we denote by Π_S the set of simple roots of $\mathfrak{k}_S$. Let also W_S be the relative Weyl group: $W_S = \langle s_\alpha \mid \alpha \in \Pi_S \rangle$, Δ_S the relative root system, $\Delta_S = W_S \Pi_S$, and θ_S the highest root of Δ_S. We recall that the dimension of $\mathfrak{k}_0$ is at most one.

We define the affine Lie algebra $\widehat{\mathfrak{k}}$ as follows. Consider the standard loop algebra $\tilde{\mathfrak{k}} = L(\mathfrak{k}) = \oplus_S L(\mathfrak{k}_S)$. On each simple ideal $\mathfrak{k}_S$ let $(\cdot,\cdot)_S$ be the normalized invariant form. Set also $(\cdot,\cdot)_0$ to be the normalized invariant form of $\mathfrak{g}$ restricted to $\mathfrak{k}_0$. We then let $\mathfrak{k}'_S = L(\mathfrak{k}_S) \oplus \mathbb{C}K_S$ be the central extension of $L(\mathfrak{k}_S)$ with bracket defined as usual as

$$[t^m \otimes X, t^n \otimes Y] = t^{n+m} \otimes [X,Y] + \delta_{m,-n} m (X,Y)_S K_S.$$

Set finally $\widehat{\mathfrak{k}} = (\oplus_S \mathfrak{k}'_S) \oplus \mathbb{C}d_{\mathfrak{k}}$, where $d_{\mathfrak{k}}$ is the derivation $t\frac{d}{dt}$ on $L(\mathfrak{k})$ extended by setting $[d_{\mathfrak{k}}, K_S] = 0$. Set $\widehat{\mathfrak{k}}_S = \mathfrak{k}'_S \oplus \mathbb{C}d_{\mathfrak{k}}$. We can extend the form $(\cdot,\cdot)_S$ on all of $\widehat{\mathfrak{k}}_S$ by setting $(K_S, \mathfrak{k}_S)_S = (K_S, K_S)_S = (d_{\mathfrak{k}}, \mathfrak{k}_S)_S = (d_{\mathfrak{k}}, d_{\mathfrak{k}})_S = 0$ and $(d_{\mathfrak{k}}, K_S)_S = 1$.

We denote by $\widehat{W}_{\mathfrak{k}}$ the Weyl group of $\widehat{\mathfrak{k}}$. It is a group of linear transformations on $\widehat{\mathfrak{h}}_{\mathfrak{k}}^*$, where

$$\widehat{\mathfrak{h}}_{\mathfrak{k}} = 1 \otimes \mathfrak{h}_0 \oplus (\oplus_S \mathbb{C}K_S) \oplus \mathbb{C}d_{\mathfrak{k}}.$$

If we define $\delta_{\mathfrak{k}} \in \widehat{\mathfrak{h}}_{\mathfrak{k}}^*$ setting

$$\delta_{\mathfrak{k}}(d_{\mathfrak{k}}) = 1, \quad \delta_{\mathfrak{k}}(1 \otimes \mathfrak{h}_0) = \delta_{\mathfrak{k}}(K_S) = 0,$$

then the set of roots for $\widehat{\mathfrak{k}}$ is

$$\widehat{\Delta}_{\mathfrak{k}} = \{n\delta_{\mathfrak{k}} + \alpha \mid \alpha \in \Delta_{\mathfrak{k}},\ n \in \mathbb{Z}\} \cup \{n\delta_{\mathfrak{k}} \mid n \in \mathbb{Z},\ n \neq 0\},$$

where, as usual, we regard $\mathfrak{h}_0^*$ as a subset of $\widehat{\mathfrak{h}}_{\mathfrak{k}}^*$ by extending $\lambda \in \mathfrak{h}_0^*$ setting $\lambda(d_{\mathfrak{k}}) = \lambda(K_S) = 0$.

We set $\Pi_{\mathfrak{k}} = \cup_S \Pi_S$ and

$$\widehat{\Pi}_{\mathfrak{k}} = \Pi_{\mathfrak{k}} \cup \{\delta_{\mathfrak{k}} - \theta_S \mid S > 0\}.$$

$\widehat{\Pi}$ is a set of simple roots for $\widehat{\mathfrak{k}}$ and we denote by $\widehat{\Delta}_{\mathfrak{k}}^+$ the corresponding subset of positive roots.

If $\lambda \in \widehat{\mathfrak{h}}_{\mathfrak{k}}^*$ is a $\widehat{\Delta}_{\mathfrak{k}}^+$-dominant integral weight, we denote by $L(\lambda)$ the irreducible integrable $\widehat{\mathfrak{k}}$ module of highest weight λ. We denote by Λ_j^S the fundamental weights of $\widehat{\mathfrak{k}}_S$ and we set $\widehat{\rho}_{\mathfrak{k}} = \sum_{j,S>0} \Lambda_j^S$. Recall the Weyl-Kac character formula for the character of $L(\lambda)$, $\lambda \in \widehat{\mathfrak{h}}_{\mathfrak{k}}^*$:

$$ch(L(\lambda)) = \frac{\sum_{w \in \widehat{W}_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda + \widehat{\rho}_{\mathfrak{k}}) - \widehat{\rho}_{\mathfrak{k}}}}{\prod_{\alpha \in \widehat{\Delta}_{\mathfrak{k}}^+} (1 - e^{-\alpha})^{m_\alpha}}. \tag{2.6}$$

Here m_α denotes the multiplicity of the root α.

2.3 Realization of level 1 modules of $\widehat{so(\mathfrak{p})}$

If $X \in \mathfrak{k}$ set $ad_{\mathfrak{p}}(X) = ad(X)_{|\mathfrak{p}}$. Since the action of $\mathfrak{k}$ on $\mathfrak{p}$ is orthogonal with respect to $(\cdot, \cdot)$, we have an inclusion $\mathfrak{k} \subset so(\mathfrak{p})$ defined by $X \mapsto ad_{\mathfrak{p}}(X)$. We let $\widehat{so(\mathfrak{p})}$ denote the affine Lie algebra $L(so(\mathfrak{p})) \oplus \mathbb{C}K_{\mathfrak{p}} \oplus \mathbb{C}d_{\mathfrak{p}}$, where $L(so(\mathfrak{p})) \oplus \mathbb{C}K_{\mathfrak{p}}$ is the central extension of $L(so(\mathfrak{p}))$ defined by setting

$$[t^m \otimes A, t^n \otimes B] = t^{n+m} \otimes [A, B] + \delta_{m,-n} m < A, B > K_{\mathfrak{p}}$$

and $< A, B >= \frac{1}{2} tr(AB)$. Note that $\widehat{so(\mathfrak{p})}$ is an affine algebra of type $B^{(1)}$ or $D^{(1)}$ according to whether $\dim(\mathfrak{p})$ is odd or even.

In the following we recall the realization of the level 1 irreducible modules of $\widehat{so(\mathfrak{p})}$ described in [13].

Fix $r \in \mathbb{Z}$, set $r' = \lfloor \frac{r}{2} \rfloor$ and consider the loop space $\tilde{\mathfrak{p}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{p}$. Define the bilinear form Φ_r on $\tilde{\mathfrak{p}}$ by setting

$$\Phi_r(t^{m_1} \otimes X, t^{m_2} \otimes Y) = \delta_{r+m_1+m_2,-1}(X, Y).$$

Let $Cl_r(\tilde{\mathfrak{p}}) = Cl_r^+(\tilde{\mathfrak{p}}) \oplus Cl_r^-(\tilde{\mathfrak{p}})$ be the corresponding Clifford algebra, decomposed into the sum of the even and odd part.

If $m \in \mathbb{Z}$ set $\tilde{\mathfrak{p}}_m = \oplus_{i \geq m}(t^i \otimes \mathfrak{p})$ and $\tilde{\mathfrak{p}}'_m = \oplus_{i < m}(t^i \otimes \mathfrak{p})$. If r is even set $\tilde{U}_r = \tilde{\mathfrak{p}}_{-r'}$. Then $\tilde{U}_r$ is a maximal isotropic subspace for $\tilde{\mathfrak{p}}$ with respect to Φ_r. If r is odd we choose a maximal isotropic subspace of $\tilde{\mathfrak{p}}$ as follows. Recall

(see [26, § 9.3.1]) that a set of positive roots Δ^+ for $\mathfrak{g}$ is *compatible with* $\Delta^+_{\mathfrak{k}}$ if it is σ-stable and $\Delta^+\cap\Delta_{\mathfrak{k}}\supseteq\Delta^+_{\mathfrak{k}}$. Let Δ^+ be such a positive system. Set $\Delta^+(\mathfrak{p})=\Delta(\mathfrak{p})\cap\Delta^+|_{\mathfrak{h}_0}$ and

$$\mathfrak{p}^{\pm}=\sum_{\alpha\in\pm\Delta^+(\mathfrak{p})}\mathfrak{p}_\alpha.$$

Thus we can write

$$\mathfrak{p}=\mathfrak{a}_{\mathbb{C}}\oplus\mathfrak{p}^+\oplus\mathfrak{p}^-,$$

where $\mathfrak{a}_{\mathbb{C}}=\mathfrak{z}\cap\mathfrak{p}$. Choose a maximal isotropic subspace $\mathfrak{a}$ of $\mathfrak{a}_{\mathbb{C}}$. Set $U=\mathfrak{a}\oplus\mathfrak{p}^+$ and

$$\tilde U_r=\tilde{\mathfrak{p}}_{-r'}\oplus(t^{-r'-1}\otimes U).$$

Let $\widehat{\sigma}_r$ denote the left action of $\widehat{so(\mathfrak{p})}$ on the spin module (defined in [13]) $s_r(\tilde{\mathfrak{p}},\tilde U_r)=Cl_r(\tilde{\mathfrak{p}})/Cl_r(\tilde{\mathfrak{p}})\tilde U_r$.

If r is even we let X_r be the subalgebra of $Cl_r(\tilde{\mathfrak{p}})$ generated by $\tilde{\mathfrak{p}}'_{-r'}$. If r is odd, set $L=\dim\mathfrak{a}_{\mathbb{C}}$ and $l=\lfloor\frac{L}{2}\rfloor=\dim\mathfrak{a}$. Fix a basis $\{v_i\}$ of $\mathfrak{a}_{\mathbb{C}}$ such that $\{v_i\mid i\le l\}$ is a basis of $\mathfrak{a}$ and $(v_i,v_{L-j+1})=\delta_{ij}$. Then, if L is even, we let X_r be the subalgebra of $Cl_r(\tilde{\mathfrak{p}})$ generated by

$$\tilde{\mathfrak{p}}'_{-r'-1}\oplus(t^{-r'-1}\otimes(Span\,(v_i\mid i>l)\oplus\mathfrak{p}^-))$$

while, if L is odd, we let $X_r=X_r^+$ be the subalgebra of $Cl_r(\tilde{\mathfrak{p}})$ generated by

$$\left(\tilde{\mathfrak{p}}'_{-r'-1}\oplus(t^{-r'-1}\otimes(Span\,(v_i\mid i>l+1)\oplus\mathfrak{p}^-))\right)(t^{-r'-1}\otimes v_{l+1}).$$

If rL is even (resp. odd) then $Cl_r(\tilde{\mathfrak{p}})=Cl_r(\tilde{\mathfrak{p}})\tilde U_r\oplus X_r$ (resp. $Cl_r^+(\tilde{\mathfrak{p}})=Cl_r^-(\tilde{\mathfrak{p}})\tilde U_r\oplus X_r$) therefore we can identify $s_r(\tilde{\mathfrak{p}},\tilde U_r)$ and X_r. Set moreover $X_r^{\pm}=X_r\cap Cl_r^{\pm}(\tilde{\mathfrak{p}})$. Set $m=\lfloor\frac{\dim(\mathfrak{p})}{2}\rfloor$ and let $\tilde\Lambda_0,\dots,\tilde\Lambda_m$ denote the fundamental weights of $\widehat{so(\mathfrak{p})}$ normalized by setting $\tilde\Lambda_i(d_{\mathfrak{p}})=0$. Define an element $\delta_{\mathfrak{p}}$ in the dual of the Cartan subalgebra of $\widehat{so(\mathfrak{p})}$ requiring that $\delta_{\mathfrak{p}}(d_{\mathfrak{p}})=1$, $\delta_{\mathfrak{p}}(K_{\mathfrak{p}})=0$, $\delta_{\mathfrak{p}}(x)=0$ for any x in the Cartan subalgebra of $so(\mathfrak{p})$.

Proposition 2.5. *[13]*

1. *The action $\widehat{\sigma}_r$ of $\widehat{so(\mathfrak{p})}$ on X_r is described explicitly in Theorem 1 of [13].*

2. *If rL is even we have $X_r^+\cong L(\tilde\Lambda_0)$, $X_r^-\cong L(\tilde\Lambda_1-\frac{1}{2}\delta_{\mathfrak{p}})$ if r is even and $X_r^+\cong L(\tilde\Lambda_m)$, $X_r^-\cong L(\tilde\Lambda_{m-1})$ if r is odd.*

3. If both L and r are odd we have $X_r^+ \cong L(\tilde{\Lambda}_m)$.

We shall conventionally refer to X_r as the *basic and vector representation* if r is even and as the *spin representation* if r is odd.

Let $\eta : \widehat{\mathfrak{k}} \to \widehat{so(\mathfrak{p})}$ be the Lie algebra homomorphism such that $t^m \otimes X \mapsto t^m \otimes ad_{\mathfrak{p}}(X)$ and $d_{\mathfrak{k}} \mapsto d_{\mathfrak{p}}$. Requiring that η is a Lie algebra homomorphism fixes the value of $\eta(K_S)$: if h is a nonzero element of $\mathfrak{h}_0 \cap \mathfrak{k}_S$, then

$$\eta([t \otimes h, t^{-1} \otimes h]) = \eta((h,h)_S K_S)$$

while

$$[\eta(t \otimes h), \eta(t^{-1} \otimes h)] = [t \otimes ad_{\mathfrak{p}}(h), t^{-1} \otimes ad_{\mathfrak{p}}(h)] = < ad_{\mathfrak{p}}(h), ad_{\mathfrak{p}}(h) > K_{\mathfrak{p}}$$

so

$$\eta(K_S) = \frac{< ad_{\mathfrak{p}}(h), ad_{\mathfrak{p}}(h) >}{(h,h)_S} K_{\mathfrak{p}}.$$

Set

$$j_S = \frac{< ad_{\mathfrak{p}}(h), ad_{\mathfrak{p}}(h) >}{(h,h)_S}.$$

Let $\kappa(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$ and $\kappa_{\mathfrak{k}}(\cdot,\cdot)$ the Killing form of $\mathfrak{k}$. We have

$$tr(ad_{\mathfrak{p}}(h) ad_{\mathfrak{p}}(h)) = \kappa(h,h) - \kappa_{\mathfrak{k}}(h,h).$$

By Corollary 8.7 of [12] we have that $\frac{\kappa_{\mathfrak{k}}(h,h)}{2(h,h)_S} = h_S^\vee$ where $h_S^\vee$ is the dual Coxeter number of $\mathfrak{k}_S$ if $S > 0$ while $h_0^\vee = 0$. If $\mathfrak{g}$ is simple we can apply Corollary 8.7 of [12] obtaining $\frac{\kappa(h,h)}{(h,h)} = 2kh^\vee$ (here $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$). It follows that

$$\frac{\kappa(h,h)}{(h,h)_S} = 2kh^\vee \cdot \frac{(h,h)}{(h,h)_S}. \tag{2.7}$$

In the complex case one checks directly that (2.7) still holds. The final outcome is that j_S is independent of the choice of h and that we can write

$$j_S = kh^\vee \cdot \frac{(h,h)}{(h,h)_S} - h_S^\vee. \tag{2.8}$$

Notice that, if $S > 0$, $\frac{(h,h)}{(h,h)_S} = \frac{2}{(\theta_S,\theta_S)}$. Setting $n_S = \frac{2k}{(\theta_S,\theta_S)} = \frac{a_0 k(\alpha_0,\alpha_0)}{(\theta_S,\theta_S)}$ for $S > 0$ and $n_0 = 1$, we see that, since $a_0 k(\alpha_0,\alpha_0)$ is the length of a long root of $\widehat{L}(\mathfrak{g},\sigma)$, we can rewrite (2.8) as

$$j_S = n_S h^\vee - h_S^\vee$$

where n_S is an integer (=1,2,3 or 4).

The map η defines a representation σ_r of $\widehat{\mathfrak{k}}$ on X_r by setting $\sigma_r = \widehat{\sigma}_r \circ \eta$. Using Theorem 1 of [13], we can describe explicitly the action σ_r of the Cartan subalgebra $\widehat{\mathfrak{h}}_{\mathfrak{k}}$ as follows. Fix weight vectors $X_\alpha \in \mathfrak{p}_\alpha$ such that $(X_\alpha, X_{-\alpha}) = 1$ and set

$$\xi_{i,\alpha} = \begin{cases} t^i \otimes X_\alpha & \text{if } rL \text{ is even} \\ \sqrt{-2}(t^i \otimes X_\alpha)(t^{-r'-1} \otimes v_{l+1}) & \text{if } rL \text{ is odd} \end{cases}.$$

Set also

$$v_{i,j} = \begin{cases} t^i \otimes v_j & \text{if } rL \text{ is even} \\ \sqrt{-2}(t^i \otimes v_j)(t^{-r'-1} \otimes v_{l+1}) & \text{if } rL \text{ is odd} \end{cases}.$$

Set

$$J_- = \{(i,\alpha) \mid i \le -r'-1,\ \alpha \in \Delta(\mathfrak{p})\} \cup \{(i,j) \mid i \le -r'-1,\ j = 1,\dots,L\}$$

if r is even, and

$$\begin{aligned} J_- =& \{(i,\alpha) \mid i < -r'-1,\ \alpha \in \Delta(\mathfrak{p})\} \cup \{(-r'-1,\alpha) \mid \alpha \in -\Delta^+(\mathfrak{p})\} \\ &\cup \{(i,j) \mid i < -r'-1,\ j = 1,\dots,L\} \cup \{(-r'-1,j) \mid L-j+1 \le l\} \end{aligned}$$

if r is odd. Putting any total order on J_-, the vectors

$$v_{i_1,j_1} \dots v_{i_h,j_h} \xi_{m_1,\beta_1} \dots \xi_{m_k,\beta_k} \tag{2.9}$$

with $(i_1,j_1) < \dots < (i_h,j_h)$ and $(m_1,\beta_1) < \dots < (m_k,\beta_k)$ in J_- form a basis for X_r. If v is a vector given by (2.9) and $h \in \mathfrak{h}_0$ then

$$\sigma_r(h)v = (\sum_{s=1}^k \beta_s(h) + \tfrac{1}{2}\delta_{r,2r'+1} \sum_{\alpha \in \Delta^+(\mathfrak{p})} \alpha(h))v$$

while

$$\sigma_r(d_{\mathfrak{k}})v = (\sum_{s=1}^h (i_s + \frac{r+1}{2}) + \sum_{s=1}^k (m_s + \frac{r+1}{2}))v.$$

Since $K_{\mathfrak{p}}$ acts as the identity on X_r we find that

$$\sigma_r(K_S)v = j_S v.$$

It follows that v is a weight vector having weight

$$\sum_S j_S \Lambda_0^S + \sum_{s=1}^h (i_s + \frac{r+1}{2})\delta_{\mathfrak{k}} + \sum_{s=1}^k ((m_s + \frac{r+1}{2})\delta_{\mathfrak{k}} + \beta_s) + \delta_{r,2r'+1}\rho_n, \tag{2.10}$$

where by definition $\rho_n = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{p})}\alpha$.

We shall use this formula in the next sections, treating separately the cases r even, r odd to obtain the decomposition of the $\widehat{so(\mathfrak{p})}$-modules X_r with respect to the subalgebra $\eta(\widehat{\mathfrak{k}})$.

3 Decomposition of the basic and vector representation (semisimple case)

Here we assume that r is even and $\mathfrak{k}$ is semisimple. Set $c_S = \frac{(h,h)}{(h,h)_S}$, where h is any nonzero element of $\mathfrak{h}_0 \cap \mathfrak{k}_S$. As already observed c_S does not depend on h. Define a linear map $\psi_0 : \widehat{\mathfrak{k}} \to \widehat{L}(\mathfrak{g},\sigma)$ by setting

$$\psi_0(t^n \otimes X) = t^{2n} \otimes X \quad \psi_0(d_{\mathfrak{k}}) = d'/2 \quad \psi_0(K_S) = 2c_S K' = kc_S K. \tag{3.1}$$

Let $\psi_0^* : \widehat{\mathfrak{h}}^* \to (\widehat{\mathfrak{h}}_{\mathfrak{k}})^*$ denote the transpose of ψ_0 (restricted to $\widehat{\mathfrak{h}}^*$). Clearly $\delta_{\mathfrak{k}} = 2\psi_0^*(\delta')$. Set

$$\widehat{\Delta}^+(\mathfrak{p}) = \{(m+1/2)\delta_{\mathfrak{k}} + \alpha \mid \alpha \in \Delta(\mathfrak{p}),\, m \geq 0\}.$$

Let us record the following facts.

Lemma 3.1.

1. *The map ψ_0^* defines a bijection between $\widehat{\Delta}^+$ and $\widehat{\Delta}^+_{\mathfrak{k}} \cup \widehat{\Delta}^+(\mathfrak{p})$.*

2. *$\psi_0^*(\widehat{\mathfrak{h}})$ is stable under the action of $\widehat{W}_{\mathfrak{k}}$.*

Proof. For the first statement, remark that $\psi_0^*(m\delta' + \alpha) = \frac{m}{2}\delta_{\mathfrak{k}} + \alpha$ for all $\alpha \in \Delta_{\mathfrak{k}} \cup \Delta(\mathfrak{p}) \cup \{0\}$. To prove the second statement, note that if $\alpha = m\delta_{\mathfrak{k}} + \beta$ with β a nonzero root in $\Delta_{\mathfrak{k}}$ and $\lambda = \psi_0^*(\mu)$, then $\alpha = \psi_0^*(2m\delta' + \beta)$, hence

$$s_\alpha\lambda = \lambda - \lambda(\alpha^\vee)\alpha = \psi_0^*(\mu - \lambda(\alpha^\vee)(2m\delta' + \beta)).$$

$\square$

Since ψ_0 is onto, ψ_0^* is bijective on its image so, for each $w \in \widehat{W}_{\mathfrak{k}}$, we can define

$$\hat{w} = (\psi_0^*)^{-1} w \psi_0^*.$$

Then we set $\widehat{W}_{\sigma,0} = \{\hat{w} \mid w \in \widehat{W}_{\mathfrak{k}}\}$.

Lemma 3.2. *Let $\alpha \in \widehat{\Delta}_{\mathfrak{k}}$ be a real root. If $\alpha = \psi_0^*(\gamma)$ then $\psi_0(\alpha^\vee) = \gamma^\vee$. In particular $\hat{s}_\alpha = s_\gamma$.*

Proof. Assume that α is a real root of $\widehat{\mathfrak{k}}_S$. We define a bilinear form $\{\cdot,\cdot\}$ on $\widehat{\mathfrak{h}}_{\mathfrak{k}}\cap\widehat{\mathfrak{k}}_S$ by setting

$$\{h,k\}=(\psi_0(h),\psi_0(k)).$$

Obviously, if $h,k\in\mathfrak{h}_0$ then $\{h,k\}=(h,k)=c_S(h,k)_S$. We claim that

$$\{\cdot,\cdot\}=c_S(\cdot,\cdot)_S.$$

Indeed, if $h\in\mathfrak{h}_0\cap\mathfrak{k}_S$, then $\{K_S,h\}=kc_S(K,h)=0$ and $\{K_S,K_S\}=k^2c_S^2(K,K)=0$. By (2.5) $\{d_{\mathfrak{k}},h\}=\frac{1}{2}(d',h)=0$, $\{d_{\mathfrak{k}},d_{\mathfrak{k}}\}=\frac{1}{4}(d',d')=0$. Finally

$$\{d_{\mathfrak{k}},K_S\}=c_S(d',K')=c_S.$$

Let $\nu_S:\widehat{\mathfrak{h}}_{\mathfrak{k}}\cap\widehat{\mathfrak{k}}_S\to(\widehat{\mathfrak{h}}_{\mathfrak{k}}\cap\widehat{\mathfrak{k}}_S)^*$ be the isomorphism induced by $\{\cdot,\cdot\}$. We claim that, if $\alpha=\psi_0^*(\gamma)\in\widehat{\Delta}_S$, then $\psi_0(\nu_S^{-1}(\alpha))=\nu^{-1}(\gamma)$. Indeed write $\nu_S^{-1}(\alpha)=h_S+aK_S$ with $h_S\in\mathfrak{k}_S\cap\mathfrak{h}_0$. If $h\in\widehat{\mathfrak{h}}$, there exists $h'\in\widehat{\mathfrak{h}}_{\mathfrak{k}}$ such that $h=\psi_0(h')$. Write $h'=\sum_i(h'_i+b_iK_i)+cd_{\mathfrak{k}}$ with $h'_i\in\mathfrak{k}_i\cap\mathfrak{h}_0$. Then

$$\begin{aligned}(\psi_0(\nu_S^{-1}(\alpha)),h)&=(h_S+kc_SaK,\sum_i(h'_i+kc_ib_iK)+\tfrac{1}{2}cd')\\&=(h_S+kc_SaK,h'_S+kc_Sb_SK+\tfrac{c}{2}d')\\&=(\psi_0(\nu_S^{-1}(\alpha)),\psi_0(h'_S+b_SK_S+cd_{\mathfrak{k}}))\\&=\{\nu_S^{-1}(\alpha),h'_S+b_SK_S+cd_{\mathfrak{k}}\}\\&=\alpha(h')=\gamma(\psi_0(h'))=\gamma(h).\end{aligned}$$

In particular

$$\psi_0(\alpha^\vee)=\psi_0(\frac{2\nu_S^{-1}(\alpha)}{\{\nu_S^{-1}(\alpha),\nu_S^{-1}(\alpha)\}})=\frac{2\nu^{-1}(\gamma)}{(\nu^{-1}(\gamma),\nu^{-1}(\gamma))}=\gamma^\vee.$$

□

Let $\Pi_0=\{\alpha_i\in\widehat{\Pi}\mid s_i=0\}$ and let $\widehat{\Pi}_{\sigma,0}=\Pi_0\cup\{k\delta-\theta_S\mid S>0\}$. Set also

$$\widehat{\Delta}_{\sigma,0}=\Delta_{\mathfrak{k}}+2\mathbb{Z}\delta' \tag{3.2}$$

An obvious consequence of Lemma 3.2 is the following fact.

Corollary 3.3. *$\widehat{\Delta}_{\sigma,0}$ is a root system and $\widehat{W}_{\sigma,0}$ is the corresponding reflection group. In particular $\widehat{W}_{\sigma,0}$ is the subgroup of $\widehat{W}$ generated by the reflections $\{s_\alpha\mid\alpha\in\widehat{\Pi}_{\sigma,0}\}$.*

Proof. For the first statement observe that $\psi_0^*(\widehat{\Delta}_{\sigma,0})$ is the set of real roots in $\widehat{\Delta}_{\mathfrak{k}}$. As for the second assertion, since $k\delta=2\delta'$, it is clear that ψ_0^* is a bijection between $\widehat{\Pi}_{\sigma,0}$ and $\widehat{\Pi}_{\mathfrak{k}}$. □

Set $N = rank(\mathfrak{g})$ and for $\alpha \in \widehat{\Delta}^+(\mathfrak{p})$ set $m_\alpha = 1$ if $\alpha = (m+\frac{1}{2})\delta_{\mathfrak{k}}+\beta$ with $\beta \in \Delta(\mathfrak{p}) \setminus \{0\}$, while we set $m_\alpha = N - n$ if $\alpha = (m + \frac{1}{2})\delta_{\mathfrak{k}}$. Observe that, if $\alpha = \psi_0^*(\beta)$, then m_α equals the multiplicity m_β of β as a root of $\widehat{L}(\mathfrak{g}, \sigma)$ (see [12], Corollary 8.3).

Let $\widehat{\rho}$ be the element of $\widehat{\mathfrak{h}}^*$ such that $\widehat{\rho}(d') = 0$ and $\widehat{\rho}(\alpha_i^\vee) = 1$ for $i = 0, \dots, n$. Set

$$D_{\mathfrak{g}}^- = e^{\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}}} \prod_{\alpha \in \widehat{\Delta}^+(\mathfrak{p})} (1 - e^{-\alpha})^{m_\alpha} \prod_{\alpha \in \widehat{\Delta}_{\mathfrak{k}}^+} (1 - e^{-\alpha})^{m_\alpha}.$$

Lemma 3.4. *We have*

$$D_{\mathfrak{g}}^- = e^{\psi_0^*(\widehat{\rho})} \prod_{\alpha \in \widehat{\Delta}^+} (1 - e^{-\psi_0^*(\alpha)})^{m_\alpha}.$$

Proof. By Lemma 3.1

$$D_{\mathfrak{g}}^- = e^{\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}}} \prod_{\alpha \in \widehat{\Delta}^+} (1 - e^{-\psi_0^*(\alpha)})^{m_\alpha}.$$

It remains only to check that

$$\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}} = \psi_0^*(\widehat{\rho}). \tag{3.3}$$

Since $\psi_0(\alpha^\vee) = \alpha^\vee$ for $\alpha \in \Pi_{\mathfrak{k}}$ we see that $\psi_0^*(\widehat{\rho})(\alpha^\vee) = 1 = (\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}})(\alpha^\vee)$ for $\alpha \in \Pi_{\mathfrak{k}}$. We defined $\widehat{\rho}$ so that $\widehat{\rho}(d') = 0$ hence $\psi_0^*(\widehat{\rho})(d_{\mathfrak{k}}) = 0 = (\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}})(d_{\mathfrak{k}})$. It remains only to check that $\psi_0^*(\widehat{\rho})(K_S) = (\sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}})(K_S) = j_S + h_S^\vee$, but, since $\psi_0^*(\widehat{\rho})(K_S) = kc_S \widehat{\rho}(K)$, this follows immediately from (2.8) and the fact that $\widehat{\rho}(K) = h^\vee$. □

By formula (2.10), the character of X_r can be written as

$$ch(X_r) = e^{\sum_S j_S \Lambda_0^S} \prod_{\alpha \in \widehat{\Delta}^+(\mathfrak{p})} (1 + e^{-\alpha})^{m_\alpha} \tag{3.4}$$

hence

$$ch(X_r^+) - ch(X_r^-) = e^{\sum_S j_S \Lambda_0^S} \prod_{\alpha \in \widehat{\Delta}^+(\mathfrak{p})} (1 - e^{-\alpha})^{m_\alpha}$$

Applying Lemma 3.4 and setting

$$D_{\mathfrak{k}} = e^{\widehat{\rho}_{\mathfrak{k}}} \prod_{\alpha \in \widehat{\Delta}_{\mathfrak{k}}^+} (1 - e^{-\alpha})^{m_\alpha},$$

we can write

$$ch(X_r^+) - ch(X_r^-) = \frac{D_{\mathfrak{g}}^-}{D_{\mathfrak{k}}} = \frac{\prod_{\alpha\in\widehat{\Delta}^+}(1-e^{-\psi_0^*(\alpha)})^{m_\alpha}}{D_{\mathfrak{k}}} \tag{3.5}$$

By the "denominator identity" (cf. [12], (10.4.4)), (3.5) can be rewritten as

$$\frac{\sum_{w\in\widehat{W}}\epsilon(w)e^{\psi_0^*(w(\widehat{\rho}))}}{D_{\mathfrak{k}}} \tag{3.6}$$

Let $W'_{\sigma,0}$ be the set of minimal right coset representatives of $\widehat{W}_{\sigma,0}$ in $\widehat{W}$. We can rewrite (3.6) as

$$\frac{\sum_{u\in W'_{\sigma,0}}\epsilon(u)\sum_{w\in\widehat{W}_{\mathfrak{k}}}\epsilon(w)e^{w\psi_0^*(u(\widehat{\rho}))-\widehat{\rho}_{\mathfrak{k}}}}{\prod_{\alpha\in\widehat{\Delta}^+_{\mathfrak{k}}}(1-e^{-\alpha})^{m_\alpha}}$$

Using the Weyl-Kac Character formula in the formulation (2.6), the final outcome is that, if r is even,

$$ch(X_r^+) - ch(X_r^-) = \sum_{u\in W'_{\sigma,0}}\epsilon(u)\,ch(L(\psi_0^*(u\widehat{\rho}) - \widehat{\rho}_{\mathfrak{k}})) \tag{3.7}$$

(cf. [23]). Using (3.3), we obtain the following result

Theorem 3.5. *If $\mathfrak{k}$ is semisimple and r is even then for $\epsilon = 0$ or 1 one has:*

$$ch(L(\tilde{\Lambda}_\epsilon)) = \sum_{\substack{u\in W'_{\sigma,0}\\ \ell(u)\equiv\epsilon\, mod\, 2}} ch\left(L(\psi_0^*(u\widehat{\rho}-\widehat{\rho}) + \sum_S j_S\Lambda_0^S + \frac{1}{2}\epsilon\delta_{\mathfrak{k}})\right), \tag{3.8}$$

where $W'_{\sigma,0}$ is the set of minimal right coset representatives of $\widehat{W}_{\sigma,0}$ in $\widehat{W}$, $\widehat{W}_{\sigma,0}$ being given by Corollary (3.3) and ψ_0 is defined by (3.1).

Proof. If λ is a weight of X_r^+ then $\lambda(d_{\mathfrak{k}})\in\mathbb{Z}$, while, if λ is a weight of X_r^- then $\lambda(d_{\mathfrak{k}})\in\frac{1}{2}+\mathbb{Z}$ (cf. Proposition 2.5). It follows that X_r^+ and X_r^- do not have common components. □

3.1 Decomposition rules and combinatorics of roots

Let Σ denote the set of $\mathfrak{b}_0$-stable abelian subspaces of $\mathfrak{p}$. Each abelian subspace in Σ is a sum of $\mathfrak{h}_0$-weight spaces. We identify $\mathfrak{i}\in\Sigma$ and the set $A\subseteq\Delta(\mathfrak{p})$ such that $\mathfrak{i}=\sum_{\alpha\in A}\mathfrak{p}_\alpha$. In this section, we describe the connection

between the subspaces in Σ and the decomposition of the basic and vector modules $L(\tilde{\Lambda}_\epsilon)$ stated in Theorem 3.5.

The set Σ has been studied in [4] in the case of $\mathfrak{g}$ simple. The results of that paper can be easily extended to the complex case. In the complex case, the subspaces in Σ turn out to correspond to more familiar objects. Indeed, we shall see that we can view Σ as the set of abelian ideals of $\mathfrak{b}_0$. We deal at once with this special case.

We recall some general conventions and facts. Let $\mathfrak{l}$ be a simple Lie algebra, Φ its root system, $\mathfrak{b}_\mathfrak{l}$ a Borel subalgebra, Φ^+ and S the corresponding set of positive roots and simple roots, respectively. A subset A of Φ^+ is called *abelian* if $\alpha+\beta\notin\Phi$ for all $\alpha,\beta\in A$. An *abelian ideal* of Φ^+ is an abelian set A such that if $\alpha\in A$ and $\gamma,\alpha+\gamma\in\Phi^+$, then $\alpha+\gamma\in A$. If A is an abelian ideal of Φ^+, then $\sum_{\alpha\in A}\mathfrak{l}_\alpha$ is an abelian ideal of $\mathfrak{b}_\mathfrak{l}$, and, conversely, each abelian ideal of $\mathfrak{b}_\mathfrak{l}$ is (uniquely) obtained in this way. Recently, there has been a great deal of work on these ideals by several authors (Kostant [17] [18], Cellini-Papi [5][6][7], Panyushev [21][22], Suter [25]). There are various explicit descriptions of them and, in particular, we know that they are exactly $2^{\mathrm{rank}(\mathfrak{l})}$.

Now let $\bar{\mathfrak{k}}$ be a simple Lie algebra, $\mathfrak{g}=\bar{\mathfrak{k}}\oplus\bar{\mathfrak{k}}$, and $\sigma:(x,y)\mapsto(y,x)$ be the switch automorphism of $\mathfrak{g}$. Thus $\mathfrak{k}$ is the diagonal copy of $\bar{\mathfrak{k}}$ in $\mathfrak{g}$, and $\mathfrak{p}=\{(x,-x)\mid x\in\bar{\mathfrak{k}}\}$. Then $\mathfrak{p}$ is naturally isomorphic to $\mathfrak{k}$ as a $\mathfrak{k}$-module, so that what we are going to study is the decomposition of the basic and vector representations of $\widehat{so(\mathfrak{k})}$ with respect to $\widehat{\mathfrak{k}}$, where $\mathfrak{k}$ is any simple Lie algebra.

Clearly, $\Delta(\mathfrak{p})=\Delta_\mathfrak{k}\cup\{0\}$ and, through the natural isomorphism between $\mathfrak{k}$ and $\mathfrak{p}$, $\mathfrak{p}_\alpha$ corresponds to $\mathfrak{k}_\alpha$ for all $\alpha\in\Delta(\mathfrak{p})$, where we intend $\mathfrak{k}_0=\mathfrak{h}_0$. In particular, if a subset A of $\Delta(\mathfrak{p})$ belongs to Σ, then by definition $\sum_{\alpha\in A}\mathfrak{p}_\alpha$ is a $\mathfrak{b}_0$ stable abelian subspace of $\mathfrak{p}$ and therefore $\sum_{\alpha\in A}\mathfrak{k}_\alpha$ is a $\mathfrak{b}_0$ stable abelian subspace of $\mathfrak{k}$. It is easily seen that this implies $A\subseteq\Delta_\mathfrak{k}^+$, and hence that A is an abelian ideal of $\Delta_\mathfrak{k}^+$.

Thus, in this case, Σ is the set of abelian ideals of $\Delta_\mathfrak{k}^+$. The following theorem, which is an easy consequence of Theorem 3.5 and the results of [5], describes the decomposition of the basic and vector representations $L(\tilde{\Lambda}_0)$ and $L(\tilde{\Lambda}_1)$ of $\widehat{so(\mathfrak{k})}$ with respect to $\widehat{\mathfrak{k}}$ in terms of Σ (cf. with [16], formula (4.2.13)). It is the nicest special case of Theorem 3.8 below.

Let us fix some notation. Set

$$\widehat{\mathfrak{h}}_\mathbb{R}=Span_\mathbb{R}(\alpha_1^\vee,\dots,\alpha_n^\vee)+\mathbb{R}K'+\mathbb{R}d'$$

and

$$\widehat{\mathfrak{h}}_1^*=\{x\in\widehat{\mathfrak{h}}_\mathbb{R}^*\mid(x,\delta)=1\},\qquad \widehat{\mathfrak{h}}_0^*=\{x\in\widehat{\mathfrak{h}}_\mathbb{R}^*\mid(x,\delta)=0\}.$$

Let π be the canonical projection mod δ and set $\mathfrak{h}_1^* = \pi\widehat{\mathfrak{h}}_1^*$. We identify $\mathfrak{h}_0^*$ with $\pi\widehat{\mathfrak{h}}_0^*$.

For $\alpha \in \widehat{\Delta}^+$ set

$$H_\alpha = \{x \in \mathfrak{h}_1^* \mid (\alpha, x) = 0\}$$

and $H_\alpha^+ = \{x \in \mathfrak{h}_1^* \mid (\alpha, x) \geq 0\}$. Also, let C_1 be the fundamental alcove of $\widehat{W}$,

$$C_1 = \{x \in \mathfrak{h}_1^* \mid (\alpha, x) \geq 0 \,\forall\, \alpha \in \widehat{\Pi}\}.$$

It is well-known that there is a faithful action of $\widehat{W}$ on $\mathfrak{h}_1^*$ and that C_1 is a fundamental domain for this action.

For $w \in \widehat{W}$ we set

$$N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in \widehat{\Delta}^-\}.$$

Finally, for $A \in \Sigma$, we denote by $\langle A \rangle$ (resp. $|A|$) the sum (resp. the number) of elements in A.

Theorem 3.6. *Let $\epsilon = 0$ or 1. Then one has the following decomposition of the basic and vector $\widehat{so(\mathfrak{k})}$-modules with respect to $\widehat{\mathfrak{k}}$.*

$$L(\tilde{\Lambda}_\epsilon) = \bigoplus_{\substack{A \in \Sigma \\ |A| \equiv \epsilon \, mod\, 2}} L(h_{\mathfrak{k}}^\vee \Lambda_0^{\mathfrak{k}} + \langle A \rangle - \frac{1}{2}(|A| - \epsilon)\delta_{\mathfrak{k}})$$

(where $h_{\mathfrak{k}}^\vee$ and $\Lambda_0^{\mathfrak{k}}$ are respectively the dual Coxeter number and 0th fundamental weight of $\widehat{\mathfrak{k}}$).

Moreover, the highest weight vector v_A of the submodule $L(h_{\mathfrak{k}}^\vee \Lambda_0^{\mathfrak{k}} + \langle A \rangle - \frac{1}{2}(|A| - \epsilon)\delta_{\mathfrak{k}})$ is, up to a constant factor, the following pure spinor (of the spin representation of $Cl_0(\tilde{\mathfrak{k}})$):

$$v_A = \prod_{\alpha \in A} (t^{-1} e_\alpha).$$

Proof. Under our assumptions, the summation $\sum_S j_S \Lambda_0^S$ in (3.5) has a single summand, which is $j_{\mathfrak{k}} \Lambda_0^{\mathfrak{k}}$. Clearly, $\widehat{L}(\mathfrak{g}, \sigma)$ is isomorphic to $\widehat{\mathfrak{k}}$, hence $h^\vee = h_{\mathfrak{k}}^\vee$. Since $k = 2$, using (2.8), we obtain that $j_{\mathfrak{k}} \Lambda_0^{\mathfrak{k}} = h_{\mathfrak{k}}^\vee \Lambda_0^{\mathfrak{k}}$.

Now we note that $\widehat{\Delta} = \mathbb{Z}^* \delta \cup (\Delta_{\mathfrak{k}} + \mathbb{Z}\delta)$, and $\widehat{\Delta}_{\mathfrak{k}} = \psi_0^*(2\mathbb{Z}^* \delta \cup (\Delta_{\mathfrak{k}} + 2\mathbb{Z}\delta))$. Hence, by Lemma 3.2, we obtain that $\widehat{W}_{\sigma,0}$ is the subgroup of $\widehat{W}$ generated by the reflections with respect to roots in $\Delta_{\mathfrak{k}} + 2\mathbb{Z}\delta$. Then $\widehat{W}_{\sigma,0}$ is isomorphic to $\widehat{W}$ itself and, moreover, it has $2C_1$ as an alcove. More precisely, $2C_1$ is the

fundamental alcove of $W_{\sigma,0}$ if we choose $\Delta_{\mathfrak{k}} + 2\mathbb{Z}\delta \cap \widehat{\Delta}^+$ as positive system for the real root system $\Delta_{\mathfrak{k}} + 2\mathbb{Z}\delta$. By general theory, it follows that

$$W'_{\sigma,0} = \{w \in \widehat{W} \mid wC_1 \subset 2C_1\}.$$

Let A be an abelian ideal of $\Delta_{\mathfrak{k}}^+$ and consider the set $-A+\delta \subset \widehat{\Delta}^+$. It is easy to prove that both $-A+\delta$ and its complement in $\widehat{\Delta}^+$ are closed under root addition, and hence that there exists a unique element $w_A \in \widehat{W}$ such that $-A+\delta = N(w_A)$. Moreover, in [5] it is proved that $A \mapsto w_A$ is a bijection between the set Σ of abelian ideals of $\Delta_{\mathfrak{k}}^+$ and the subset $\{w \in \widehat{W} \mid wC_1 \subset 2C_1\}$. Now the claim follows directly from Theorem 3.5 and the following observations:

1. for $w \in \widehat{W}$, $w(\widehat{\rho}) - \widehat{\rho} = -\langle N(w)\rangle$ (see e.g. [19], Corollary 1.3.22);
2. for $\alpha \in \widehat{\Delta}^+$, $\psi_0^*(-\alpha+\delta) = -\alpha + \frac{\delta_{\mathfrak{k}}}{2}$;
3. for $A \in \Sigma$, $\epsilon(w_A) = (-1)^{|A|}$, and $|A| = |-A+\delta| = |N(w_A)| = \ell(w_A)$.

The statement on highest weight vectors will be proved in Theorem 3.9. □

We now turn to the combinatorial interpretation of the decomposition (3.8) for general $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k}$ semisimple. We need to recall some results and notation from [4].

If $\mathfrak{g}$ is simple, by the classification of Lie algebra involutions (see [12], Ch.8), we have that there exists an index p such that $s_p = 1$ and $s_i = 0$ if $i \neq p$.

Set

$$D_\sigma = \bigcup_{w \in \mathcal{W}_{ab}^\sigma} wC_1, \tag{3.9}$$

where

$$\mathcal{W}_{ab}^\sigma = \left\{ w \in \widehat{W} \mid N(w) \subseteq \{\alpha \in \widehat{\Delta} \mid m_p(\alpha) = 1\} \right\}.$$

and, as above, $N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in \widehat{\Delta}^-\}$. If $w \in \mathcal{W}_{ab}^\sigma$ we shall say that w is *σ-minuscule*.

Given $w \in \widehat{W}$, a root $\beta \in \widehat{\Delta}^+$ belongs to $N(w)$ if and only if H_β separates wC_1 and C_1. It follows that

$$D_\sigma = \bigcap_{\substack{\alpha \in \widehat{\Delta}^+, \\ m_p(\alpha) \neq 1}} H_\alpha^+.$$

If $\alpha \in \widehat{\Delta}$ we set $\widehat{H}_\alpha = \{x \in \widehat{\mathfrak{h}}_{\mathbb{R}}^*/\mathbb{R}\delta \mid (\alpha, x) = 0\}$ and $\widehat{H}_\alpha^+ = \{x \in \widehat{\mathfrak{h}}_{\mathbb{R}}^*/\mathbb{R}\delta \mid (\alpha, x) \geq 0\}$. Set also

$$C_\sigma = \bigcap_{\substack{\alpha \in \widehat{\Delta}^+, \\ m_p(\alpha) \neq 1}} \widehat{H}_\alpha^+.$$

Obviously

$$D_\sigma = C_\sigma \cap \mathfrak{h}_1^*.$$

Set $\Phi_\sigma = \widehat{\Pi}_{\sigma,0} \cup \{k_p\delta + \alpha_p\}$, where $k_p = k$, 1 according to whether α_p is long or short.

Proposition 3.7. *[4] We have that*

$$C_\sigma = \bigcap_{\alpha \in \Phi_\sigma} \widehat{H}_\alpha^+.$$

Now let

$$P_\sigma = \bigcap_{\alpha \in \widehat{\Pi}_{\sigma,0}} H_\alpha^+. \tag{3.10}$$

It is a standard fact that the set of elements $w \in \widehat{W}$ such that wC_1 cover the polytope P_σ is the set of minimal right coset representatives for the subgroup of $\widehat{W}$ generated by s_α with $\alpha \in \widehat{\Pi}_{\sigma,0}$, which, by Corollary 3.3, happens to be $\widehat{W}_{\sigma,0}$. Since obviously $D_\sigma \subseteq P_\sigma$, we have that $\mathcal{W}_{ab}^\sigma \subset W'_{\sigma,0}$. We elucidate the precise relation between $\mathcal{W}_{ab}^\sigma$ and $W'_{\sigma,0}$ in the next proposition where, if $\widehat{L}(\mathfrak{g}, \sigma)$ is simply laced, we regard all roots as long. Recall that we denote by $\mathfrak{b}_0$ the Borel subalgebra of $\mathfrak{k}$ associated to our initial choice of positive roots in $\Delta_{\mathfrak{k}}$.

Recall that we are assuming that $\mathfrak{k}$ is semisimple. If $\mathfrak{g}$ is simple, let p be the index such that $s_p = 1$ and $s_i = 0$ if $i \neq p$. In the complex case (and only in this case) we have $p = 0$ (see Remark 2.2). For $\gamma \in Q^\vee$, we denote by t_γ the translation by γ (see [12, (6.5.2)]).

Proposition 3.8. *[5],[4] 1). $D_\sigma = P_\sigma$ if and only if $p = 0$ or α_p is short. If α_p is a long root, then $P_\sigma \setminus D_\sigma$ consists exactly of the alcove $w_\sigma C_1$, where $w_\sigma = t_{-k\alpha_p^\vee} w_0^* w_0$, w_0 is the longest element of the parabolic subgroup of $\widehat{W}$ generated by $\widehat{\Pi} \setminus \{\alpha_p\}$ and w_0^* is the longest element of the parabolic subgroup of $\widehat{W}$ generated by $\widehat{\Pi} \cap \alpha_p^\perp$.*
2). There is a bijection between $\mathfrak{b}_0$-stable abelian subspaces in $\mathfrak{p}$ and σ-minuscule elements, or, equivalently, alcoves paving D_σ. In this correspondence an element $w \in \mathcal{W}_{ab}^\sigma$ such that $N(w) = \{\beta_1, \ldots, \beta_r\}$ maps to $\bigoplus_{i=1}^r \mathfrak{p}_{-\overline{\beta_i}}$.

Set $\Lambda_{0,\mathfrak{k}} = \sum j_S \Lambda_0^S$ and let Σ denote the set of $\mathfrak{b}_0$-stable abelian subspaces of $\mathfrak{p}$. Identify $\mathfrak{i} \in \Sigma$ and the set $A \subset \Delta(\mathfrak{p})$ such that $\mathfrak{i} = \sum_{\alpha \in A} \mathfrak{p}_\alpha$. We summarize the connection between abelian subspaces and the decomposition of X_r in the following proposition.

Theorem 3.9. *(1) Assume that $p = 0$ or α_p is a short root. Then*

$$L(\tilde{\Lambda}_\epsilon) = \bigoplus_{\substack{A\in\Sigma \\ |A|\equiv \epsilon \ mod\, 2}} L\left(\Lambda_{0,\mathfrak{k}} + \langle A\rangle - \frac{1}{2}(|A| - \epsilon)\delta_{\mathfrak{k}}\right).$$

(2) Assume that α_p is a long root and $p \neq 0$. We have

$$L(\tilde{\Lambda}_\epsilon) = \bigoplus_{\substack{A\in\Sigma \\ |A|\equiv \epsilon \ mod\, 2}} L\left(\Lambda_{0,\mathfrak{k}} + \langle A\rangle - \frac{1}{2}(|A| - \epsilon)\delta_{\mathfrak{k}})\right) \bigoplus \nu L\left(\Lambda_{0,\mathfrak{k}} - y + \frac{1}{2}\epsilon\delta_{\mathfrak{k}}\right),$$

where

$$y := \psi_0^*(\langle N(w_\sigma)\rangle) = (\sum_{\beta\in(\alpha_p+\Delta_{\mathfrak{k}}^+)\cap\widehat{\Delta}^+} \overline{\beta}) + 2\overline{\alpha}_p + \left(\frac{|(\alpha_p + \Delta_{\mathfrak{k}}^+)\cap\widehat{\Delta}^+|}{2} + 2\right)\delta_{\mathfrak{k}}$$

and $\nu = \delta_{\epsilon,\ell(w_\sigma)\, mod\, 2}$.

Moreover, in both cases, the highest weight vector of each component is, up to a constant factor, the pure spinor (of the spin representation of $Cl_r(\tilde{\mathfrak{p}})$):

$$v_A = \prod_{\alpha\in A}(t^{-r'-2}e_\alpha) \tag{3.11}$$

where $\mathfrak{p}_\alpha = \mathbb{C}e_\alpha$. A highest weight vector in the component indexed by w_σ is

$$v_\sigma = \prod_{\beta\in(\alpha_p+\Delta_{\mathfrak{k}}^+)\cap\widehat{\Delta}^+}(t^{-r'-2}e_{-\overline{\beta}})(t^{-r'-2}e_{-\overline{\alpha}_p})(t^{-r'-3}e_{-\overline{\alpha}_p}). \tag{3.12}$$

Proof. It follows immediately from (3.3) that

$$\psi_0^*(u\widehat{\rho}) - \widehat{\rho}_{\mathfrak{k}} = \Lambda_{0,\mathfrak{k}} - \psi_0^*(\langle N(u)\rangle). \tag{3.13}$$

hence we can rewrite formulas (3.8) as

$$ch(X_r) = ch(L(\tilde{\Lambda}_0)) + ch(L(\tilde{\Lambda}_1)) = \sum_{w\in W'_{\sigma,0}} ch(L(\Lambda_{0,\mathfrak{k}} - \psi_0^*(\langle N(w)\rangle))).$$

By Corollary 3.3 and Proposition 3.8, we can write

$$ch(X_r) = \sum_{w \in \mathcal{W}_{ab}^{\sigma}} ch(L(\Lambda_{0,\mathfrak{k}} - \psi_0^*(\langle N(w) \rangle)))$$

if $p = 0$ or α_p is short, while

$$ch(X_r) = \sum_{w \in \mathcal{W}_{ab}^{\sigma} \cup \{w_\sigma\}} ch(L(\Lambda_{0,\mathfrak{k}} - \psi_0^*(\langle N(w) \rangle)))$$

if $p \neq 0$ and α_p is long. If $\alpha \in \widehat{\Delta}$ then $\psi_0^*(\alpha) = \frac{1}{2} m_p(\alpha)\delta_{\mathfrak{k}} + \overline{\alpha}$. If $w = w_A$ for some $A \in \Sigma$ and $\alpha \in N(w_A)$ then $m_p(\alpha) = 1$. Moreover, if $w_A \in \mathcal{W}_{ab}^{\sigma}$ encodes the subspace A, we have $\epsilon(w_A) = (-1)^{\ell(w_A)} = (-1)^{|A|}$. This justifies the distribution of the summands in the basic and vector modules according to the parity of $|A|$.

The calculation of $N(w_\sigma)$ follows by a straightforward computation using standard properties of the sets $N(w)$ (see [7], 2.5). One gets

$$N(w_\sigma) = \left(\alpha_p + \Delta_{\mathfrak{k}}^+\right) \cap \widehat{\Delta}^+ \cup \{\alpha_p\} \cup \{\alpha_p + k\delta\}. \tag{3.14}$$

Since $k\delta = k \sum_{i=0}^n a_i s_i \delta = 2\delta'$ if we apply ψ_0^* to each element in the r.h.s. of (3.14) and take the sum we obtain the required expression for y.

We now check that v_A is a highest weight vector. Set $\lambda_A = \psi_0^*(w_A(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}}$ be the corresponding highest weight. We will show that, if $\alpha \in \widehat{\Pi}_{\mathfrak{k}}$, then $\lambda_A + \alpha$ is not a weight of X_r. Indeed $\lambda_A + \widehat{\rho}_{\mathfrak{k}} = \psi_0^*(w_A(\widehat{\rho}))$ and $\alpha = \psi_0^*(\beta)$ with $\beta \in \widehat{\Pi}_{\sigma,0}$ so we can write $\lambda_A + \widehat{\rho}_{\mathfrak{k}} + \alpha = \psi_0^*(w_A(\widehat{\rho}) + \beta)$. We observe that

$$(w_A(\widehat{\rho}) + \beta, w_A(\widehat{\rho}) + \beta) = (\widehat{\rho}, \widehat{\rho}) + (\beta, \beta) + 2(w_A(\widehat{\rho}), \beta).$$

Since $w_A(C_1) \subset P_\sigma$, we have that $2(w_A(\widehat{\rho}), \beta) \geq 0$, hence

$$(w_A(\widehat{\rho}) + \beta, w_A(\widehat{\rho}) + \beta) > (\widehat{\rho}, \widehat{\rho}).$$

If $\lambda_A + \alpha$ is a weight of X_r, then, according to (3.4), $\lambda_A + \alpha + \widehat{\rho}_{\mathfrak{k}} \in \sum_S j_S \Lambda_0^S + \widehat{\rho}_{\mathfrak{k}} - \psi_0^*(S)$, where S is the set of weights defined in Lemma 3.2.3 of [19], thus we can write that $\psi_0^*(w_A(\widehat{\rho}) + \beta) \in \psi_0^*(\widehat{\rho} - S)$. It follows that $w_A(\widehat{\rho}) + \beta - \widehat{\rho} \in -S$. Applying Lemma 3.2.4 of [19] (with $\mu = w_A(\widehat{\rho}) + \beta - \widehat{\rho}$), we find a contradiction. Obviously the same argument applies also to w_σ. $\square$

4 Decomposition of the spin representation (semisimple case)

We now consider the case when r is odd and $\mathfrak{k}$ is semisimple. We distinguish two cases: $\mathfrak{g}$ not simple (the complex case) and $\mathfrak{g}$ simple.

4.1 Complex case

We consider here the case when $\mathfrak{g} = \mathfrak{k} \times \mathfrak{k}, \sigma(X, Y) = (Y, X)$, $\mathfrak{k}$ is simple and embeds in $\mathfrak{g}$ diagonally. We have that $\Delta(\mathfrak{p}) = \Delta_{\mathfrak{k}} \cup \{0\}$ and we can choose $\Delta^+(\mathfrak{p}) = \Delta^+_{\mathfrak{k}}$. In this case $\mathfrak{k}$ is simple, so, by (2.8), the sum $\sum_S j_S \Lambda_{0,S}$ reduces to one summand, which equals $\widehat{\rho}_{\mathfrak{k}}$. By (2.10) the character of X_r is

$$\begin{aligned}
&ch(X_r) = \\
&= e^{\widehat{\rho}_{\mathfrak{k}}} 2^{\lfloor \frac{n}{2} \rfloor} \left(\prod_{\alpha \in \widehat{\Delta}^+_{\mathfrak{k}}} (1 + e^{-\alpha})^{m_\alpha} \right) = e^{\widehat{\rho}_{\mathfrak{k}}} 2^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\prod_{\alpha \in \widehat{\Delta}^+_{\mathfrak{k}}} (1 - e^{-2\alpha})^{m_\alpha} \right)}{\left(\prod_{\alpha \in \widehat{\Delta}^+_{\mathfrak{k}}} (1 - e^{-\alpha})^{m_\alpha} \right)} \\
&= e^{\widehat{\rho}_{\mathfrak{k}}} 2^{\lfloor \frac{n}{2} \rfloor} \frac{\sum_{w \in \widehat{W}} \epsilon(w) e^{2w\widehat{\rho}_{\mathfrak{k}} - 2\widehat{\rho}_{\mathfrak{k}}}}{\left(\prod_{\alpha \in \widehat{\Delta}^+_{\mathfrak{k}}} (1 - e^{-\alpha})^{m_\alpha} \right)} = 2^{\lfloor \frac{n}{2} \rfloor} \frac{\sum_{w \in \widehat{W}} \epsilon(w) e^{w(\widehat{\rho}_{\mathfrak{k}} + \widehat{\rho}_{\mathfrak{k}}) - \widehat{\rho}_{\mathfrak{k}}}}{\left(\prod_{\alpha \in \widehat{\Delta}^+_{\mathfrak{k}}} (1 - e^{-\alpha})^{m_\alpha} \right)} \\
&= 2^{\lfloor \frac{n}{2} \rfloor} L(\widehat{\rho}_{\mathfrak{k}}).
\end{aligned}$$

Thus, we obtain

Proposition 4.1. *([16], 4.2.2). In the complex case the spin representation of $\mathfrak{k} \times \mathfrak{k}$ restricts to $2^{\lfloor \frac{rk(\mathfrak{k})}{2} \rfloor}$ copies of the $\widehat{\mathfrak{k}}$-module $L(\widehat{\rho}_{\mathfrak{k}})$.*

4.2 $\mathfrak{g}$ simple case

We assume now that r is odd and $\mathfrak{k}$ is a semisimple symmetric subalgebra of a simple algebra $\mathfrak{g}$.

Structure theory. By the classification of Lie algebra involutions (see [12], Ch.8), we have that there exists $p \in \{0, \ldots, n\}$ such that $ka_p = 2$ and $s_p = 1$ while $s_i = 0$ for $i \neq p$. Set ϖ_p to be the unique element of $\mathfrak{h}_0$ such that $\overline{\alpha}_i(\varpi_p) = \delta_{ip}$ for $i = 1, \ldots, n$. Set

$$\mu = \sigma \circ \exp(\pi i ad(\varpi_p)).$$

Let $\mathfrak{k}_\mu$ denote the set of μ-fixed points in $\mathfrak{g}$.

It is easy to show that $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{k}_\mu$: if $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{k}_\mu$ containing $\mathfrak{h}_0$ then $[\mathfrak{h}', \varpi_p] = 0$, so $\mathfrak{h}' \subset \mathfrak{k}$. This implies $\mathfrak{h}' = \mathfrak{h}_0$. If m is a positive integer such that $\sigma^m = \mu^m = id$, then, by Proposition 8.5 of [12] (with notation therein) the map $t^{\frac{m}{2}\varpi_p}$ is an isomorphism $L(\mathfrak{g}, \mu, m) \to L(\mathfrak{g}, \sigma, m)$. In particular the linear map $t_p : \overline{\alpha} + i\delta' \mapsto \overline{\alpha} - \frac{m}{2}(i - \overline{\alpha}(\varpi_p))\delta'$ defines a bijection between $\widehat{\Delta}$ and the set of $\widehat{\mathfrak{h}}$-roots of $\widehat{L}(\mathfrak{g}, \mu, m)$. It follows that $t_p(\widehat{\Pi})$ is a set of simple roots for $t_p(\widehat{\Delta}^+)$.

If $z \in \mathbb{Z}$, set $L(\mathfrak{g},\mu,m)_z = \{x \in L(\mathfrak{g},\mu,m) \mid d'\cdot x = zx\}$. Since $t_p(\alpha_i) = \overline{\alpha}_i$ if $i>0$, $t_p(\alpha_0) = \frac{m}{ka_0}\delta' + \overline{\alpha}_0$ and $\mathfrak{k}_\mu = L(\mathfrak{g},\mu,m)_0$, we see that the set of $\mathfrak{h}_0$-roots of $\mathfrak{k}_\mu$ is $\Delta_f := \{\overline{\alpha} \mid \alpha \in \widehat{\Delta},\, m_0(\alpha)=0\}$.

Clearly $\Pi_f = \{\overline{\alpha}_1,\dots,\overline{\alpha}_n\}$ is a set of simple roots for Δ_f and the corresponding set of positive roots is $\Delta_f^+ = \{\overline{\alpha} \mid \alpha \in \widehat{\Delta}^+,\, m_0(\alpha)=0\}$.

Explicit description of $\Delta_{\mathfrak{k}}$ and $\Delta(\mathfrak{p})$. Set

$$\Delta_f^0 = \{\alpha \in \Delta_f \mid, \alpha(\varpi_p) \equiv 0 \mod 2\},\ \Delta_f^1 = \{\alpha \in \Delta_f \mid \alpha(\varpi_p) \equiv 1 \mod 2\}$$

and let $\Delta_{f,s}$ and $\Delta_{f,l}$ be, respectively, the set of short and long roots in Δ_f. We let $\Delta_x^\epsilon = \Delta_x \cap \Delta_f^\epsilon$ ($x = f,l$ or f,s; $\epsilon = 0,1$).

Recall from Section 2 our classification of $\mathfrak{z}$-roots of $\mathfrak{g}$ into complex, compact, and noncompact roots. Set

$$\begin{aligned}
\Delta_{cx} &= \{\alpha \in \Delta(\mathfrak{p}) \mid \alpha = \beta_{|\mathfrak{h}_0},\ \beta \text{ complex}\},\\
\Delta_{ci} &= \{\alpha \in \Delta_{\mathfrak{k}} \mid \alpha = \beta_{|\mathfrak{h}_0},\ \beta \text{ compact}\},\\
\Delta_{ni} &= \{\alpha \in \Delta(\mathfrak{p}) \mid \alpha = \beta_{|\mathfrak{h}_0},\ \beta \text{ noncompact}\};\\
\widehat{\Delta}_{cx} &= \{i\delta_{\mathfrak{k}} + \alpha \mid i \in \mathbb{Z},\, \alpha \in \Delta_{cx}\},\\
\widehat{\Delta}_{ci} &= \{i\delta_{\mathfrak{k}} + \alpha \mid i \in \mathbb{Z},\, \alpha \in \Delta_{ci}\},\\
\widehat{\Delta}_{ni} &= \{i\delta_{\mathfrak{k}} + \alpha \mid i \in \mathbb{Z},\, \alpha \in \Delta_{ni}\}.
\end{aligned}$$

If $k=1$ then $\mathfrak{z} = \mathfrak{h}_0$ and σ is of inner type. It follows that $\sigma = \exp(\pi i h)$ for some $h \in \mathfrak{h}_0$. Since $\sigma(X_j) = e^{\pi i \overline{\alpha}_j(h)}X_j = e^{\pi i s_j}X_j$ for $j = 1\dots,n$, we see that $\sigma = \exp(\pi i ad(\varpi_p))$ and $\mu = id$. Hence, in this case,

$$\Delta_{cx} = \emptyset, \quad \Delta(\mathfrak{p}) = \Delta_f^1 = \Delta_{ni}, \quad \Delta_{\mathfrak{k}} = \Delta_f^0 = \Delta_{ci}. \tag{4.1}$$

Suppose now that $k=2$, so that $\delta' = \delta$. Recall from 2.1 that $\alpha \in \Delta$ is a noncompact root if and only if $\delta + \alpha_{|\mathfrak{h}_0}$ is a long root of $\widehat{\Delta}$, α is compact if and only if $\alpha_{|\mathfrak{h}_0}$ is a long root of $\widehat{\Delta}$, and α is complex if and only if $\alpha_{|\mathfrak{h}_0} \in \widehat{\Delta}$ and it is not a long root.

Assume that $k=2$ and $\widehat{L}(\mathfrak{g},\sigma)$ is not of type $A_{2n}^{(2)}$. The following relations hold.

$$\Delta_{|\mathfrak{h}_0} = (\Delta(\mathfrak{p}) \setminus \{0\}) \cup \Delta_{\mathfrak{k}} = \overline{\widehat{\Delta}} \setminus \{0\} = \Delta_f$$

The first equality is clear, the second depends on the fact that $\widehat{\Delta}$ is the set of roots of $\widehat{L}(\mathfrak{g},\sigma)$, whereas the third follows from the explicit description of $\widehat{\Delta}$ given in Proposition 6.3 a) of [12]. From the above discussion it follows that

$$\Delta_{cx} = \Delta_{f,s}, \quad \Delta_{ci} = \Delta_{f,l}^0, \quad \Delta_{ni} = \Delta_{f,l}^1. \tag{4.2}$$

Moreover

$$\Delta(\mathfrak{p}) = \Delta_{ni} \cup \Delta_{cx} = \Delta^1_{f,l} \cup \Delta_{f,s}, \qquad \Delta_{\mathfrak{k}} = \Delta_{ci} \cup \Delta_{cx} = \Delta^0_{f,l} \cup \Delta_{f,s}.$$

If $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A^{(2)}_{2n}$, then $\Delta_{\mathfrak{k}}$ is the subsystem of $\widehat{\Delta}$ generated by $\{\alpha_0, \dots, \alpha_{n-1}\}$. It follows that $\Delta_{\mathfrak{k}}$ does not contain long roots of $\widehat{\Delta}$, hence $\Delta_{\mathfrak{k}} = \Delta_{cx}$. Since $\overline{\widehat{\Delta}} = \Delta_f \cup \frac{1}{2}\Delta_{f,l} \cup \{0\}$ (see again [12], Prop. 6.3 b)), arguing as above we have

$$\Delta_{ni} = \Delta_{f,l} \quad \Delta_{cx} = \Delta_{\mathfrak{k}} = \tfrac{1}{2}\Delta_{f,l} \cup \Delta_{f,s}. \tag{4.3}$$

As we have seen in Section 2.3, the explicit realization of the spin module depends on the choice of a set of positive roots Δ for $\mathfrak{g}$ that is compatible with $\Delta^+_{\mathfrak{k}}$. We make a particular choice that we now explain.

Let u be the longest element in the Weyl group of $\mathfrak{k}$, u' the longest element in the parabolic subgroup corresponding to $\Pi_{\mathfrak{k}} \setminus \{\alpha_0\}$, and $w_0 = uu'$. Clearly w_0 stabilizes both $\Delta_{\mathfrak{k}}$ and $\Delta(\mathfrak{p})$, hence $\Delta_{|\mathfrak{h}_0} \subset w_0(\frac{1}{2}\Delta_f \cup \Delta_f)$. It is easy to see that $\Delta^+_{\mathfrak{k}} \subset w_0(\frac{1}{2}\Delta^+_f \cup \Delta^+_f)$. It follows that

$$\Delta^+ = \{\alpha \in \Delta \mid \alpha_{|\mathfrak{h}_0} \in w_0(\tfrac{1}{2}\Delta^+_f \cup \Delta^+_f)\}$$

is a positive set of roots for Δ compatible with $\Delta^+_{\mathfrak{k}}$. Recall that we set $\Delta^+(\mathfrak{p}) = \Delta^+_{|\mathfrak{h}_0} \cap \Delta(\mathfrak{p})$. We let

$$\Delta^+_{cx} = \Delta_{cx} \cap \Delta^+_{\mathfrak{k}}, \quad \Delta^+_{ci} = \Delta_{ci} \cap \Delta^+_{\mathfrak{k}}, \quad \Delta^+_{ni} = \Delta_{ni} \cap \Delta^+(\mathfrak{p})$$
$$\widehat{\Delta}^+_a = \Delta^+_a \cup \{j\delta_{\mathfrak{k}} + \alpha \mid j > 0,\, \alpha \in \Delta_a\} \quad (a = cx, ci, ni).$$

The algebras $L'(\mathfrak{g}, \sigma)$. Recall that $(\cdot,\cdot)_n$ denotes the normalized invariant form on $\mathfrak{g}$. Since there is $\overline{\alpha}_i \in \Delta_f$ such that α_i is long in $\widehat{\Delta}$, it follows that $(\cdot,\cdot)_{n|\mathfrak{k}_\mu}$ is the normalized invariant form on $\mathfrak{k}_\mu$. If Δ_f is a root system of type Y_n, we can realize the affine Lie algebra of type $Y^{(1)}_n$ as the subalgebra $\widehat{\mathfrak{k}}_\mu = L(\mathfrak{k}_\mu) \oplus \mathbb{C}K' \oplus \mathbb{C}d'$ of $\widehat{L}(\mathfrak{g})$. We set $(\cdot,\cdot) = (\cdot,\cdot)_n$ in (2.1), so that K' is the canonical central element of $\widehat{\mathfrak{k}}_\mu$. We denote by $\widehat{\Delta}_\mu$ the set of roots of $\widehat{\mathfrak{k}}_\mu$ with respect to $\widehat{\mathfrak{h}}$ and by $\widehat{W}_{\mathfrak{k}_\mu}$ its Weyl group. If $\overline{\theta}_f$ is the highest root of Δ_f with respect to Π_f, then $\widehat{\Pi}_\mu = \{-\overline{\theta}_f + \delta', \overline{\alpha}_1, \dots, \overline{\alpha}_n\}$ is a set of simple roots of $\widehat{\mathfrak{k}}_\mu$ with respect to $\widehat{\mathfrak{h}}$. With this choice of the simple roots, the set of positive roots is

$$\widehat{\Delta}^+_\mu = \Delta^+_f \cup ((\Delta_f \cup \{0\}) + \mathbb{Z}^+\delta').$$

Let Λ_μ be the linear functional on $\widehat{\mathfrak{h}}$ which maps K' to 1 and $\mathfrak{h}_0 + \mathbb{C}d'$ to 0.

Let $(\cdot,\cdot)^\mu$ be the normalized invariant form of $\widehat{\mathfrak{k}}_\mu$ such that $(\Lambda_\mu,\Lambda_\mu)^\mu = 0$ and let $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ be the isomorphism induced by $(\cdot,\cdot)^\mu$. For any subset R of real roots in $\widehat{\Delta}_\mu$ we set $R^\vee = \{\nu(\alpha^\vee) \mid \alpha \in R\}$. Then it is clear that, if A is the generalized Cartan matrix of $\widehat{\mathfrak{k}}_\mu$, then $(\widehat{\mathfrak{h}}, \widehat{\Pi}_\mu^\vee, \nu^{-1}(\widehat{\Pi}_\mu))$ is a realization of the Cartan matrix ${}^tA$. Let $\widehat{\mathfrak{k}}_\mu^\vee = \mathfrak{g}({}^tA)$ be the twisted affine algebra corresponding to the given realization of ${}^tA$. By general theory of root systems, the set of real roots of $\widehat{\mathfrak{k}}_\mu^\vee$ is $\widehat{\Delta}_{\mu,re}^\vee$, where $\widehat{\Delta}_{\mu,re}$ is the set of real roots of $\widehat{\mathfrak{k}}_\mu$. Since $(\cdot,\cdot)^\mu$ is a normalized form on $\widehat{\mathfrak{k}}_\mu$, we have that the set of imaginary roots for $\widehat{\mathfrak{k}}_\mu^\vee$ is $\mathbb{Z}^*\delta'$. It follows that the set of roots of $\widehat{\mathfrak{k}}_\mu^\vee$ is $\widehat{\Delta}_\mu^\vee := \widehat{\Delta}_{\mu,re}^\vee \cup \mathbb{Z}^*\delta'$. Observe that if $\widehat{L}(\mathfrak{g},\sigma)$ is of type $X_N^{(2)} = A_{2l-1}^{(2)}, D_{l+1}^{(2)}, E_6^{(2)}$, then $\widehat{\mathfrak{k}}_\mu^\vee$ is of type $X_{N'}^{(2)} = D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}$ respectively (see [12], 13.9). Moreover, since $\nu(\alpha^\vee) = \frac{2}{(\alpha,\alpha)^\mu}\alpha$, the Weyl group of $\widehat{\mathfrak{k}}_\mu^\vee$ is $\widehat{W}_{\mathfrak{k}_\mu}$.

Remark 4.1. If $rk(\mathfrak{g}) = N$, then the number of short roots in $\Pi_f^\vee$ is $2n-N$, therefore, as a root of $\widehat{\mathfrak{k}}_\mu^\vee$, $j\delta'$ has multiplicity $2n-N$ if j is odd and n if j is even.

We define the Lie algebra $L'(\mathfrak{g},\sigma)$ as follows

$$L'(\mathfrak{g},\sigma) = \begin{cases} \widehat{\mathfrak{k}}_\mu & \text{if } k=1,\\ \widehat{\mathfrak{k}}_\mu^\vee & \text{if } k=2 \text{ and } a_0=1,\\ \widehat{L}(\mathfrak{g},\sigma)^\vee & \text{if } k=a_0=2.\end{cases}$$

In the last case $\widehat{L}(\mathfrak{g},\sigma)$ is of type $A_{2n}^{(2)}$ and $\widehat{L}(\mathfrak{g},\sigma)^\vee$ is realized with a construction analogous to that performed for $\widehat{\mathfrak{k}}_\mu^\vee$, using the normalized invariant form of $\widehat{L}(\mathfrak{g},\sigma)$. In particular, the set of roots of $L'(\mathfrak{g},\sigma)$ is

$$(\tfrac{1}{2}\Delta_{f,l} + \tfrac{1}{2}(2\mathbb{Z}-1)\delta') \cup (\Delta_{f,s} + \mathbb{Z}\delta') \cup (\Delta_{f,l} + (2\mathbb{Z})\delta') \cup \mathbb{Z}^*\delta'. \tag{4.4}$$

We will denote by $\widehat{\Delta}'$ the set of roots of $L'(\mathfrak{g},\sigma)$ in all cases. We choose $(\widehat{\Delta}')^+ = (\frac{1}{2}\widehat{\Delta}_\mu^+ \cup \widehat{\Delta}_\mu^+) \cap \widehat{\Delta}'$ as a set of positive roots and notice that the corresponding set $\widehat{\Pi}'$ of simple roots is $\widehat{\Pi}_\mu$, $\widehat{\Pi}_\mu^\vee$, and $\{\frac{1}{2}(\delta' - \overline{\theta}_f), \overline{\alpha}_1, \dots, \overline{\alpha}_n\}$ if $a_0k = 1$, 2, and 4 respectively. Let $\widehat{\rho}'$ denote the sum of the fundamental weights of $L'(\mathfrak{g},\sigma)$. Observe that the Weyl group of $L'(\mathfrak{g},\sigma)$ is $\widehat{W}_{\mathfrak{k}_\mu}$.

The map $\psi_1 : \widehat{\mathfrak{h}}_\mathfrak{k} \to \widehat{\mathfrak{h}}$. We already observed that $(\cdot,\cdot)_{n|\mathfrak{k}_\mu} = (\cdot,\cdot)^\mu_{|\mathfrak{k}_\mu}$. Also recall that we let $c_S = \frac{(h,h)}{(h,h)_S}$, where h is any nonzero element of $\mathfrak{h}_0 \cap \mathfrak{k}_S$. It follows from the discussion preceding Lemma 2.4 that

$$(h,h)^\mu = kc_S(h,h)_S.$$

Consider the linear map $\phi : \widehat{\mathfrak{h}}_{\mathfrak{k}} \to \widehat{\mathfrak{h}}$ defined by

$$\phi_{|\mathfrak{h}_0} = id_{\mathfrak{h}_0}, \quad \phi(d_{\mathfrak{k}}) = d', \quad \phi(K_S) = kc_S K'.$$

We define

$$\psi_1 = \phi \circ w_0^{-1} \tag{4.5}$$

so that

$$\psi_1^* = w_0 \circ \phi^*.$$

It is clear that ψ_1 is surjective, hence ψ_1 is injective. We denote by ψ_1^{*-1} the inverse of $\psi_1^* : \widehat{\mathfrak{h}}^* \to \psi_1^*(\widehat{\mathfrak{h}}^*)$.

It is immediate from the definition of ψ_1 that

$$\psi_1^*(\Lambda_\mu) = \sum_S kc_S \Lambda_0^S, \tag{4.6}$$

$$\psi_1^*(\delta') = \delta_{\mathfrak{k}}, \tag{4.7}$$

$$\psi_1^*(\lambda) = w_0(\lambda) \text{ for } \lambda \in \mathfrak{h}_0^*. \tag{4.8}$$

Note that, by (4.7), (4.8) and relation $w_0(\Delta_{\mathfrak{k}}) \subseteq \Delta_{\mathfrak{k}}$ we have that $\widehat{\Delta}_{\mathfrak{k}} \subset \psi_1^*(\widehat{\mathfrak{h}}^*)$, hence $\psi_1^*(\widehat{\mathfrak{h}}^*)$ is $\widehat{W}_{\mathfrak{k}}$-stable.

Lemma 4.2. *For $\alpha \in \widehat{\Delta}_{\mathfrak{k}}$, let β be the unique element of $\widehat{\Delta}_\mu$ such that $\psi_1^*(\beta)$ is a multiple of α. Let $s_\alpha : \widehat{\mathfrak{h}}_{\mathfrak{k}}^* \to \widehat{\mathfrak{h}}_{\mathfrak{k}}^*$ be the reflection with respect to α and $s'_\beta : \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*$ the reflection with respect to β. Then*

$$\psi_1^{*-1} s_\alpha \psi_1^* = s'_\beta.$$

Proof. The proof is the same as for Lemma 3.2. □

Remark 4.2. We set $\widehat{W}_{\sigma,1} = (\psi_1^*)^{-1} \widehat{W}_{\mathfrak{k}} \psi_1^*$. Lemma 4.2 says that $\widehat{W}_{\sigma,1}$ is a subgroup of $\widehat{W}_{\mathfrak{k}_\mu}$.

If $\widehat{L}(\mathfrak{g}, \sigma)$ is not of type $A_{2n}^{(2)}$ then, by Lemma 4.2, $\widehat{W}_{\sigma,1}$ is generated by the reflection s_α with α a real root in $\psi_1^{*-1}(\widehat{\Delta}_{\mathfrak{k}})$. By (4.7), and (4.1)–(4.2), we have that the set of real roots in $\psi_1^{*-1}(\widehat{\Delta}_{\mathfrak{k}})$ is

$$\widehat{\Delta}_{\sigma,1} := (\Delta_{cx} \cup \Delta_{ci}) + \mathbb{Z}\delta'. \tag{4.9}$$

If $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A_{2n}^{(2)}$, then, by Lemma 4.2 and (4.3), we have that $\widehat{W}_{\sigma,1}$ is the subgroup of $\widehat{W}_{\mathfrak{k}_\mu}$ generated by all reflections with respect to roots in

$$\widehat{\Delta}_{\sigma,1} := (\Delta_{f,s} + \mathbb{Z}\delta') \cup (\Delta_{f,l} + 2\mathbb{Z}\delta'). \tag{4.10}$$

The character. We set

$$\widehat{\Delta}^+_{re}(\mathfrak{p}) = \Delta^+(\mathfrak{p}) \cup \{\alpha + j\delta_{\mathfrak{k}} \mid \alpha \in \Delta(\mathfrak{p}) \setminus \{0\},\ j \in \mathbb{Z}^+\}.$$

From (2.10) we obtain directly

$$ch(X_r) = 2^{\lfloor \frac{N-n}{2} \rfloor} e^{\sum_S j_S \Lambda_0^S + \rho_n} \prod_{j>0} (1 + e^{-j\delta_{\mathfrak{k}}})^{N-n} \prod_{\alpha \in \widehat{\Delta}^+_{re}(\mathfrak{p})} (1 + e^{-\alpha}). \tag{4.11}$$

Recall that

$$D_{\mathfrak{k}} = e^{\widehat{\rho}_{\mathfrak{k}}} \prod_{i>0} (1 - e^{-i\delta_{\mathfrak{k}}})^n \prod_{\alpha \in (\widehat{\Delta}^+_{\mathfrak{k}})_{re}} (1 - e^{-\alpha}),$$

and set

$$\rho^* = \sum_S j_S \Lambda_0^S + \rho_n + \widehat{\rho}_{\mathfrak{k}}.$$

Then dividing and multiplying (4.11) by $D_{\mathfrak{k}}$ yields

$$ch(X_r) = 2^{\lfloor \frac{N-n}{2} \rfloor} D^+_{\mathfrak{g}} / D_{\mathfrak{k}}, \tag{4.12}$$

where

$$\begin{aligned} D^+_{\mathfrak{g}} = e^{\rho^*} & \prod_{i>0} (1 + e^{-i\delta_{\mathfrak{k}}})^{N-n} \prod_{i>0} (1 - e^{-i\delta_{\mathfrak{k}}})^n \\ & \times \prod_{\alpha \in \widehat{\Delta}^+_{ni}} (1 + e^{-\alpha}) \prod_{\alpha \in \widehat{\Delta}^+_{cx}} (1 - e^{-2\alpha}) \prod_{\alpha \in \widehat{\Delta}^+_{ci}} (1 - e^{-\alpha}). \end{aligned}$$

If $\widehat{L}(\mathfrak{g}, \sigma)$ is not of type $A^{(2)}_{2n}$ set

$$\begin{aligned} D^-_{\mathfrak{g}} = e^{\rho^*} & \prod_{i>0} (1 + e^{-i\delta_{\mathfrak{k}}})^{N-n} \prod_{i>0} (1 - e^{-i\delta_{\mathfrak{k}}})^n \\ & \times \prod_{\alpha \in \widehat{\Delta}^+_{ni} \cup \widehat{\Delta}^+_{ci}} (1 - e^{-\alpha}) \prod_{\alpha \in \widehat{\Delta}^+_{cx}} (1 - e^{-2\alpha}). \end{aligned}$$

Observe that $D^-_{\mathfrak{g}}$ differs from $D^+_{\mathfrak{g}}$ just in the product over $\widehat{\Delta}^+_{ni}$.

If $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A^{(2)}_{2n}$ then set

$$\begin{aligned} \widehat{\Delta}^{even}_{ni} &= \Delta^+_{ni} \cup (\Delta_{ni} + 2\mathbb{Z}^+ \delta_{\mathfrak{k}}) \\ \widehat{\Delta}^{odd}_{ni} &= \widehat{\Delta}^+_{ni} \setminus \widehat{\Delta}^{even}_{ni} \\ \widehat{\Delta}^+_{f,cx} &= (\Delta^+_{cx} \cap \Delta_f) \cup (\Delta_{cx} \cap \Delta_f + \mathbb{Z}^+ \delta_{\mathfrak{k}}). \end{aligned}$$

Recalling that in this case $N = 2n$ and that, by (4.3), $\Delta_{cx} = \frac{1}{2}\Delta_{ni} \cup (\Delta_{cx} \cap \Delta_f)$, we can rewrite $D^+_{\mathfrak{g}}$ as

$$\begin{aligned}
D^+_{\mathfrak{g}} &= e^{\rho^*} \prod_{i>0}(1+e^{-i\delta_{\mathfrak{k}}})^n \prod_{i>0}(1-e^{-i\delta_{\mathfrak{k}}})^n \\
&\quad \times \prod_{\alpha\in\widehat{\Delta}^{odd}_{ni}}(1+e^{-\alpha}) \prod_{\alpha\in\widehat{\Delta}^{even}_{ni}}(1+e^{-\alpha}) \prod_{\alpha\in\widehat{\Delta}^{even}_{ni}}(1-e^{-\alpha}) \prod_{\alpha\in\widehat{\Delta}^{+}_{f,cx}}(1-e^{-2\alpha}) \\
&= e^{\rho^*} \prod_{i>0}(1-e^{-2i\delta_{\mathfrak{k}}})^n \prod_{\alpha\in\widehat{\Delta}^{odd}_{ni}}(1+e^{-\alpha}) \prod_{\alpha\in\widehat{\Delta}^{even}_{ni}}(1-e^{-2\alpha}) \prod_{\alpha\in\widehat{\Delta}^{+}_{f,cx}}(1-e^{-2\alpha}).
\end{aligned}$$

In this case we set

$$D^-_{\mathfrak{g}} = e^{\rho^*} \prod_{i>0}(1-e^{-2i\delta_{\mathfrak{k}}})^n \prod_{\alpha\in\widehat{\Delta}^{odd}_{ni}}(1-e^{-\alpha}) \prod_{\alpha\in\widehat{\Delta}^{even}_{ni}}(1-e^{-2\alpha}) \prod_{\alpha\in\widehat{\Delta}^{+}_{f,cx}}(1-e^{-2\alpha}),$$

that differs from $D^+_{\mathfrak{g}}$ just in the product over $\widehat{\Delta}^{odd}_{ni}$.

First we show how to compute $D^-_{\mathfrak{g}}$ and then we shall compute $D^+_{\mathfrak{g}}$ from $D^-_{\mathfrak{g}}$.

Lemma 4.3.

$$\rho^* = \psi_1^*(a_0\widehat{\rho}')$$

Proof. We start from the equal rank case. In this case $\widehat{\rho}' = h^\vee\Lambda_\mu + \rho$, where ρ is half the sum of the roots in Δ^+_f. It follows that $\psi_1^*(\widehat{\rho}') = h^\vee\psi_1^*(\Lambda_\mu) + \psi_1^*(\rho)$. Since

$$\psi_1^*(\rho) = w_0(\rho) = \tfrac{1}{2}\sum_{\alpha\in\Delta^+_{\mathfrak{k}}}\alpha + \tfrac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{p})}\alpha = \rho_{\mathfrak{k}} + \rho_n$$

we can write that $\psi_1^*(\widehat{\rho}') = h^\vee\psi_1^*(\Lambda_\mu) + \rho_n + \rho_{\mathfrak{k}}$. By (4.6), $\psi_1^*(\Lambda_\mu) = \sum c_S\Lambda_0^S$ Hence, by (2.8), we conclude that $\psi_1^*(\widehat{\rho}') = \sum j_S\Lambda_0^S + \rho_n + \widehat{\rho}_{\mathfrak{k}}$ as desired.

If $k = 2$ and $\widehat{L}(\mathfrak{g},\sigma)$ is not of type $A^{(2)}_{2n}$, denoting by $(h')^\vee$ the dual Coxeter number of $L'(\mathfrak{g},\sigma)$, we have $\widehat{\rho}' = (h')^\vee\Lambda_\mu + \rho^\vee$, where $\rho^\vee$ is half the sum of the roots in $(\Delta^+_f)^\vee$ and in turn $\psi_1^*(\widehat{\rho}') = (h')^\vee\psi_1^*(\Lambda_\mu) + \psi_1^*(\rho^\vee)$. Since

$$\psi_1^*(\rho^\vee) = w_0(\rho^\vee) = \tfrac{1}{2}\sum_{\alpha\in\Delta^+_{ni}}\alpha + \tfrac{1}{2}\sum_{\alpha\in\Delta^+_{ci}}\alpha + \sum_{\alpha\in\Delta^+_{cx}}\alpha = \rho_{\mathfrak{k}} + \rho_n$$

we need only to check that $\psi_1^*((h')^\vee\Lambda_\mu) = \sum_S(j_S + h_S^\vee)\Lambda_0^S$. But $\psi_1^*((h')^\vee\Lambda_\mu) = \sum_S(h')^\vee 2c_S\Lambda_0^S$ which equals, by (2.8), $(h')^\vee\sum_S\frac{j_s+h_S^\vee}{h^\vee}\Lambda_0^S$. The claim follows because $h^\vee = (h')^\vee$.

Finally, if $\widehat{L}(\mathfrak{g},\sigma)$ is of type $A_{2n}^{(2)}$, we have $2\widehat{\rho}' = h^\vee\Lambda_\mu + 2\rho$. Now, from (4.3), we obtain that

$$\begin{aligned}
\psi_1^*(2\rho) &= \sum_{\alpha\in\Delta_{f,s}^+}\alpha + \sum_{\alpha\in\Delta_{f,l}^+}\alpha \\
&= \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{f,l}^+}\alpha + \sum_{\alpha\in\frac{1}{2}\Delta_{f,l}^+}\alpha + \sum_{\alpha\in\Delta_{f,s}^+}\alpha\Big) \\
&+ \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{f,s}^+}\alpha + \sum_{\alpha\in\frac{1}{2}\Delta_{f,l}^+}\alpha\Big) \\
&= \rho_n + \rho_{\mathfrak{k}}.
\end{aligned}$$

Finally, by (2.8), $\psi_1^*(h^\vee\Lambda_\mu) = h^\vee\sum_S 2c_S\Lambda_0^S = h^\vee\sum_S \frac{j_S+h_S^\vee}{h^\vee}\Lambda_0^S$ and we conclude as above. $\square$

Proposition 4.4.

$$D_{\mathfrak{g}}^- = e^{\psi_1^*(a_0\widehat{\rho}')}\prod_{\alpha\in(\widehat{\Delta}')^+}(1-e^{-\psi_1^*(a_0\alpha)})^{m_\alpha},$$

where m_α is the multiplicity of α as a root of $L'(\mathfrak{g},\sigma)$.

Proof. If $\widehat{L}(\mathfrak{g},\sigma)$ is not of type $A_{2n}^{(2)}$, then formulas (4.6)–(4.8) imply that ψ_1^* is a bijection between the set $(\widehat{\Delta}'_{re})^+$ of positive real roots in $\widehat{\Delta}'$ and $\widehat{\Delta}_{ni}^+\cup\widehat{\Delta}_{ci}^+\cup 2\widehat{\Delta}_{cx}^+$. Hence

$$D_{\mathfrak{g}}^- = e^{\rho^*}\prod_{i>0}(1+e^{-i\delta_{\mathfrak{k}}})^{N-n}\prod_{i>0}(1-e^{-i\delta_{\mathfrak{k}}})^n\prod_{\alpha\in(\widehat{\Delta}'_{re})^+}(1-e^{-\psi_1^*(\alpha)}).$$

Next we observe that

$$\begin{aligned}
\prod_{i>0}(1+e^{-i\delta_{\mathfrak{k}}})^{N-n}\prod_{i>0}(1-e^{-i\delta_{\mathfrak{k}}})^n &= \prod_{i>0}(1-e^{-2i\delta_{\mathfrak{k}}})^{N-n}\prod_{i>0}(1-e^{-i\delta_{\mathfrak{k}}})^{2n-N} = \\
&\prod_{i>0}(1-e^{-2i\delta_{\mathfrak{k}}})^{n}\prod_{i>0}(1-e^{-(2i-1)\delta_{\mathfrak{k}}})^{2n-N}.
\end{aligned} \tag{4.13}$$

hence, using Remark 4.1,

$$D_{\mathfrak{g}}^- = e^{\rho^*}\prod_{\alpha\in(\widehat{\Delta}')^+}(1-e^{-\psi_1^*(\alpha)})^{m_\alpha}.$$

Applying Lemma 4.3, we obtain the result in this case.

If $\widehat{L}(\mathfrak{g},\sigma)$ is of type $A_{2n}^{(2)}$, then, by (4.4), ψ_1^* defines a bijection between $2(\widehat{\Delta}'_{re})^+$ and $\widehat{\Delta}_{ni}^{odd}\cup 2\widehat{\Delta}_{ni}^{even}\cup 2\widehat{\Delta}^+_{f,cx}$ hence

$$D_{\mathfrak{g}}^- = e^{\rho^*}\prod_{i>0}(1-e^{-2i\delta_{\mathfrak{k}}})^n\prod_{\alpha\in(\widehat{\Delta}'_{re})^+}(1-e^{-2\psi_1^*(\alpha)}).$$

By Remark 4.1, $m(j\delta')=n$ for all j, hence

$$D_{\mathfrak{g}}^- = e^{\rho^*}\prod_{\alpha\in(\widehat{\Delta}')^+}(1-e^{-2\psi_1^*(\alpha)})^{m_\alpha}.$$

Lemma 4.3 implies the result in this case too. □

Applying Weyl-Kac denominator formula we readily obtain

Corollary 4.5.

$$D_{\mathfrak{g}}^- = e^{\rho^*}\sum_{w\in\widehat{W}_{\mathfrak{k}_\mu}}\epsilon(w)e^{a_0(\psi_1^*(w(\widehat{\rho}')-\widehat{\rho}'))}.$$

Decomposition of X_r. We now show how to compute $D_{\mathfrak{g}}^+$ from $D_{\mathfrak{g}}^-$. This will allow us to compute the decomposition of X_r.

For $\gamma_1,\dots,\gamma_t\in a_0\psi_1^*((\widehat{\Delta}')^+)$, we set

$$\epsilon_p(\gamma_1,\dots,\gamma_t)=\begin{cases}(-1)^{|\{\gamma_1,\dots,\gamma_t\}\cap\widehat{\Delta}_{ni}^{odd}|} & \text{if } \widehat{L}(\mathfrak{g},\sigma) \text{ is of type } A_{2n}^{(2)}\\ (-1)^{|\{\gamma_1,\dots,\gamma_t\}\cap\widehat{\Delta}_{ni}^{+}|} & \text{otherwise.}\end{cases}\tag{4.14}$$

Set

$$h_\sigma=\begin{cases}d_{\mathfrak{k}} & \text{if } \widehat{L}(\mathfrak{g},\sigma) \text{ is of type } A_{2n}^{(2)}\\ \varpi_p & \text{otherwise.}\end{cases}$$

By the explicit description of $a_0\psi_1^*((\widehat{\Delta}')^+)$ given in the proof of Proposition 4.4 it is clear from (4.1)–(4.3) that $(-1)^{(\gamma_1+\cdots+\gamma_t)(h_\sigma)}=\epsilon_p(\gamma_1,\dots,\gamma_t)$. In particular, if we define a function ϵ_p on the $\mathbb{Z}$-lattice L generated by $a_0\psi_1^*((\widehat{\Delta}')^+)$ by setting

$$\epsilon_p(\lambda)=(-1)^{\lambda(h_\sigma)}.$$

then, if $\lambda=\gamma_1+\cdots+\gamma_t$ with $\gamma_i\in a_0\psi_1^*((\widehat{\Delta}')^+)$, we have

$$\epsilon_p(\lambda)=\epsilon_p(\gamma_1,\dots,\gamma_t).\tag{4.15}$$

Lemma 4.6. *We have*

$$D_{\mathfrak{g}}^- = e^{\rho^*} \sum_{\lambda \in L} a_\lambda e^\lambda,$$

with $a_\lambda \in \mathbb{Z}$. Moreover

$$D_{\mathfrak{g}}^+ = e^{\rho^*} \sum_{\lambda \in L} \epsilon_p(\lambda) a_\lambda e^\lambda.$$

Proof. By Corollary 4.5

$$D_{\mathfrak{g}}^- = e^{\rho^*} \sum_{w \in \widehat{W}_{\mathfrak{k}_\mu}} \epsilon(w) e^{-a_0(\psi_1^*(<N'(w)>))},$$

where $N'(w) = \{\alpha \in (\widehat{\Delta}')^+ \mid w^{-1}(\alpha) < 0\}$, hence the first assertion follows. The second statement follows directly from (4.15) and the definition of $D_{\mathfrak{g}}^+$ and $D_{\mathfrak{g}}^-$. $\square$

We set $\widehat{\Delta}'_{\sigma,1} = \widehat{\Delta}_{\sigma,1}$ (see (4.9)) if $k = 1$ or $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A_{2n}^{(2)}$, while we set $\widehat{\Delta}'_{\sigma,1} = (\widehat{\Delta}_{\sigma,1})^\vee$ in the other cases. We notice that $\widehat{\Delta}'_{\sigma,1}$ is a root system contained in $\widehat{\Delta}'$ and its associated reflection group is $\widehat{W}_{\sigma,1}$. By general theory of reflection groups (see [8]) the set

$$W'_{\sigma,1} = \{u \in \widehat{W}_{\mathfrak{k}_\mu} \mid N'(u) \subseteq \widehat{\Delta}' \setminus \widehat{\Delta}'_{\sigma,1}\}$$

is a set of minimal coset representatives of $\widehat{W}_{\sigma,1} \backslash \widehat{W}_{\mathfrak{k}_\mu}$.

For $w \in \widehat{W}_{\mathfrak{k}_\mu}$ set $N^*(w) = N'(w) \cap \widehat{\Delta}'_{\sigma,1}$. Set also $\ell(w) = |N'(w)|$ and, if $v \in \widehat{W}_{\sigma,1}$, $\ell^*(v) = |N^*(v)|$. Now assume $v \in \widehat{W}_{\sigma,1}$ and $u \in W'_{\sigma,1}$. Since $v(\widehat{\Delta}'_{\sigma,1}) = \widehat{\Delta}'_{\sigma,1}$, we have that $vN'(u) \subseteq \widehat{\Delta}' \setminus \widehat{\Delta}'_{\sigma,1}$. It is a standard fact that $N'(v) \subset \widehat{\Delta}'_{\sigma,1}$. In particular $N'(vu) = N'(v) \,\dot\cup\, (vN'(u) \cap (\Delta'_\mu)^+)$ (disjoint union), whence, $N'(vu) \cap \widehat{\Delta}'_{\sigma,1} = N'(v) \cap \widehat{\Delta}'_{\sigma,1} = N^*(v)$.

If ϵ and ϵ^* denote the sign functions in $\widehat{W}_{\mathfrak{k}_\mu}$ and $\widehat{W}_{\sigma,1}$, respectively, then $\epsilon(w) = (-1)^{\ell(w)}$ and $\epsilon^*(v) = (-1)^{\ell^*(v)}$. Notice that the set of real roots in $\widehat{\Delta}' \setminus \widehat{\Delta}'_{\sigma,1}$ maps under $a_0\psi_1^*$ bijectively onto $\widehat{\Delta}_{ni}^{odd}$ if $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A_{2n}^{(2)}$, and onto $\widehat{\Delta}_{ni}$ in all the other cases, therefore

$$\epsilon(vu) = \epsilon^*(v)\epsilon_p(a_0\psi_1^*\langle N'(vu)\rangle). \tag{4.16}$$

It follows from Corollary 4.5 and (4.16) that

$$D_{\mathfrak{g}}^- = e^{\rho^*} \sum_{v \in \widehat{W}_{\sigma,1}} \sum_{u \in W'_{\sigma,1}} \epsilon^*(v)\epsilon_p(a_0\psi_1^*\langle N'(vu)\rangle) e^{-a_0\psi_1^*(\langle N'(vu)\rangle)},$$

and, by Lemma 4.6,

$$D^+_{\mathfrak{g}} = e^{\rho^*} \sum_{v\in \widehat{W}_{\sigma,1}} \sum_{u\in W'_{\sigma,1}} \epsilon^*(v) e^{-a_0\psi_1^*\langle N'(vu)\rangle}.$$

Clearly, if $v \in \widehat{W}_{\mathfrak{k}}$, then $\epsilon^*(\psi_1^{*-1}v\psi_1^*) = det(\psi_1^{*-1}v\psi_1^*) = det(v)$. Therefore from the above equation we obtain

$$D^+_{\mathfrak{g}} = \sum_{v\in \widehat{W}_{\sigma,1}} \sum_{u\in W'_{\sigma,1}} \epsilon^*(v) e^{a_0\psi_1^*(vu\widehat{\rho}')} = \sum_{u\in W'_{\sigma,1}} \sum_{v\in \widehat{W}_{\mathfrak{k}}} det(v) e^{v(a_0\psi_1^*(u\widehat{\rho}'))},$$

and, since $ch(X_r) = 2^{\lfloor \frac{N-n}{2}\rfloor}\frac{D^+_{\mathfrak{g}}}{D_{\mathfrak{k}}}$, from (2.6) we deduce the following result.

Proposition 4.7. *If $\mathfrak{k}$ is semisimple and r is odd, then*

$$ch(X_r) = 2^{\lfloor \frac{N-n}{2}\rfloor} \sum_{u\in W'_{\sigma,1}} ch(L(a_0\psi_1^*(u\widehat{\rho}') - \widehat{\rho}_{\mathfrak{k}})), \tag{4.17}$$

where ψ_1 is defined by (4.5).

4.3 Combinatorial interpretation of decompositions of spin modules.

We will use the following general facts. Let $\widehat{\mathfrak{g}}_1, \widehat{\mathfrak{g}}_2$ be two affine Kac-Moody algebras, and for $i = 1, 2$, let $\widehat{\mathfrak{h}}_i$ be a Cartan subalgebra of $\widehat{\mathfrak{g}}_i$, $\widehat{\Delta}_i$ be the corresponding root system, and $\widehat{W_i}$ be the Weyl group. Endow $\widehat{\mathfrak{g}}_i$, with a fixed arbitrary invariant form and $\widehat{\mathfrak{h}}_i^*$ with the form induced by this choice. We say that a linear isomorphism $f : \widehat{\mathfrak{h}}_1^* \to \widehat{\mathfrak{h}}_2^*$ is an extended isomorphism of root systems if f is an isometry and $f\widehat{\Delta}_1 = \widehat{\Delta}_2$. For $\alpha \in \widehat{\Delta}_i$ let s_α be the reflection on $\widehat{\mathfrak{h}}_i$ with respect to α. The following result is clear.

Lemma 4.8. *If f is an extended isomorphism of root systems, then, for all $\alpha \in \widehat{\Delta}_1$, $fs_\alpha f^{-1} = s_{f\alpha}$. In particular, if $A \subset \widehat{\Delta}_1$, and W_A is the subgroup of $\widehat{W}_1$ generated by the reflections with respect to elements of A, then fW_Af^{-1} is the subgroup of $\widehat{W}_2$ generated by the reflections with respect to elements in fA.*

Lemma 4.9. *Let $f : \mathfrak{h}_1^*\oplus\mathbb{C}\delta_1 \to \mathfrak{h}_2^*\oplus\mathbb{C}\delta_2$ be a linear map such that $f\widehat{\Delta}_1 = \widehat{\Delta}_2$ and, for all $\alpha, \beta \in \widehat{\Delta}_1$, $(\alpha, \beta)_1 = (f\alpha, f\beta)_2$. Then there exists a unique extension of f to an extended isomorphism of root systems.*

Proof. Since $\mathbb{C}\delta_2$ is the orthogonal subspace of $\mathfrak{h}_2^* \oplus \mathbb{C}\delta_2$ in $\widehat{\mathfrak{h}}_2^*$, and since $\widehat{\Delta}_2$ spans $\mathfrak{h}_2^* \oplus \mathbb{C}\delta_2$, the conditions $(f\Lambda_0^1, f\alpha)_2 = (\Lambda_0^1, \alpha)_1$ for all $\alpha \in \widehat{\Delta}_1$ determine $f\Lambda_0^1$ modulo $\mathbb{C}\delta_2$. By a direct computation we see that the further condition $(f\Lambda_0^1, f\Lambda_0^1)_2 = 0$ determines the component in $\mathbb{C}\delta_2$ of $f\Lambda_0^1$. □

Definition 4.3. Let us say that a $\mathfrak{h}_0$-stable subspace S of $\mathfrak{p}$ is *noncompact* if all weights of $\mathfrak{h}_0$ on S are in Δ_{ni}.

We will describe the decomposition of X_r in terms of certain noncompact subspaces of $\mathfrak{p}$. For the sake of a better exposition we discuss various cases separately: we consider the equal rank case, the case when $\widehat{L}(\mathfrak{g}, \sigma)$ is of type $A_{2n}^{(2)}$ and the remaining non equal rank cases.

Equal rank case. In the equal rank case all $\mathfrak{h}_0$-stable subspaces of $\mathfrak{p}$ are noncompact, for $\Delta(\mathfrak{p})$ is equal to Δ_{ni}, henceforth the final outcome will be very similar to decomposition of the basic and vector representations.

Recall that in this case $\mu = Id$, so $L'(\mathfrak{g}, \sigma) = \widehat{L}(\mathfrak{g})$ and $\widehat{\Delta}' = \widehat{\Delta}_\mu$. The isomorphism $t^{\varpi_p} : L(\mathfrak{g}, \mu, 2) \to L(\mathfrak{g}, \sigma)$ induces a linear isomorphism $g : \mathfrak{h}_0^* + \mathbb{C}\delta' \to \mathfrak{h}_0^* + \mathbb{C}\delta'$ such that $g(\widehat{\Delta}_\mu) = \widehat{\Delta}$. Explicitly

$$g : \lambda + j\delta' \mapsto \lambda + (2j + \lambda(\varpi_p))\delta'. \tag{4.18}$$

By (2.4) it is clear that g preserves scalar products of roots.

By (4.1), $\Delta_{\mathfrak{k}} = \Delta_f^0$ so $g(\widehat{\Delta}_{\sigma,1}) = \Delta_f^0 + 2\mathbb{Z}\delta'$. Comparing this with (3.2) we see that

$$g(\widehat{\Delta}_{\sigma,1}) = \widehat{\Delta}_{\sigma,0}. \tag{4.19}$$

By Lemma 4.8, $\widehat{W} = g\widehat{W}_{\mathfrak{k}_\mu}g^{-1}$ and, by (4.19), $g\widehat{W}_{\sigma,1}g^{-1} = \widehat{W}_{\sigma,0}$. Recall that in this case we have that $p > 0$ and $a_p = 2$, hence $g(\delta' - \overline{\theta}) = -\overline{\theta} = \overline{\alpha}_0 = \alpha_0$ and $g(\overline{\alpha}_i) = \alpha_i$ for $i = 1, \dots, n$. It follows that $g(\widehat{\Delta}_\mu^+) = \widehat{\Delta}^+$ hence $N(gug^{-1}) = g(N'(u))$, for all $u \in \widehat{W}_{\mathfrak{k}_\mu}$. It follows that

$$W'_{\sigma,1} = g^{-1}W'_{\sigma,0}\, g.$$

and $g(\widehat{\rho}') = \widehat{\rho}$.

Recalling that $N = n$ in this case, we can rewrite the decomposition of

X_r given in (4.17) as

$$\begin{aligned} X_r &= \sum_{u\in W'_{\sigma,1}} L(\sum_S j_S\Lambda_0^S + \rho_n + \psi_1^*(u(\widehat{\rho}') - \widehat{\rho}')) \\ &= \sum_{u\in W'_{\sigma,1}} L(\sum_S j_S\Lambda_0^S + \rho_n - \psi_1^*(N'(u))) \\ &= \sum_{u\in W'_{\sigma,0}} L(\sum_S j_S\Lambda_0^S + \rho_n - \psi_1^*(g^{-1}N(u))) \end{aligned}$$

Applying the discussion of 3.1 we deduce the analog of Theorem 3.9 for the spin representation in the equal rank case:

Theorem 4.10. *Set* $m = \lfloor \frac{dim(\mathfrak{p})}{2} \rfloor$.
(1) Assume that α_p *is a short root. Then*

$$L(\tilde\Lambda_{m-\epsilon}) = \bigoplus_{\substack{A\in\Sigma \\ |A|\equiv\epsilon\ mod\, 2}} L\left(\Lambda_{0,\mathfrak{k}} + \rho_n + w_0\langle A\rangle\right).$$

(2) Assume that α_p *is a long root. We have*

$$\begin{aligned} L(\tilde\Lambda_{m-\epsilon}) = &\bigoplus_{\substack{A\in\Sigma \\ |A|\equiv\epsilon\ mod\, 2}} L\left(\Lambda_{0,\mathfrak{k}} + \rho_n + w_0\langle A\rangle - k_A\delta_{\mathfrak{k}})\right) \\ &\bigoplus \nu L\left(\Lambda_{0,\mathfrak{k}} + \rho_n - w_0(y) - (k_y+1)\delta_{\mathfrak{k}}\right), \end{aligned}$$

where y *and* ν *are as in Theorem 3.9 (2) and* $k_A = |w_0(A)\cap\Delta^+(\mathfrak{p})|$, $k_y = |(\alpha_p + \Delta^+_{\mathfrak{k}}) \cap (\delta' - \Delta^+_f)|$.

Moreover, in both cases, the highest weight vector of each component indexed by $A\in\Sigma$ *is, up to a constant factor, the pure spinor (of the spin representation of* $Cl_r(\tilde{\mathfrak{p}})$*):*

$$v_A = \prod_{\alpha\in w_0(A)\cap\Delta^+(\mathfrak{p})} (t^{-r'-2}e_\alpha) \prod_{\alpha\in w_0(A)\cap(-\Delta^+(\mathfrak{p}))} t^{-r'-1}e_\alpha, \qquad (4.20)$$

where $\mathfrak{p}_\alpha = \mathbb{C}e_\alpha$*. An highest weight vector for the component indexed by* w_σ *is*

$$\begin{aligned} &(\textstyle\prod_{\beta\in(\alpha_p+\Delta^+_{\mathfrak{k}})\cap(\delta'-\Delta^+_f)} t^{-r'-2}e_{-\overline{\beta}}) \qquad (4.21) \\ &\quad (\prod_{\beta\in(\alpha_p+\Delta^+_{\mathfrak{k}})\cap(\delta'+\Delta^+_f)} t^{-r'-1}e_{-\overline{\beta}})(t^{-r'-1}e_{-\overline{\alpha}_p})(t^{-r'-2}e_{-\overline{\alpha}_p}). \end{aligned}$$

Proof. First of all observe that a weight vector v is in X_r^+ if and only if its weight is equal to $\sum_S j_S\Lambda_0^S + \rho_n + \lambda$, where λ is a sum of an even number of elements of $\widehat{\Delta}_{ni}$, hence $L(\sum_S j_S\Lambda_0^S - \psi_1^*(g^{-1}N(u)))$ occurs in X_r^+ if and only if $\ell(u)$ is even.

The rest of the result now follows as in Theorem 3.9. Only the coefficient of $\delta_{\mathfrak{k}}$ needs checking. If $A \in \Sigma$ and $\alpha \in N(w_A)$, then $m_p(\alpha) = 1$ hence $\alpha = \delta' \pm \overline{\alpha}$ with $\overline{\alpha} \in \Delta_f^1 \cap \Delta_f^+$. If $\alpha = \overline{\alpha} + \delta'$ then $g^{-1}(\alpha) = \overline{\alpha}$, while, if $\alpha = -\overline{\alpha} + \delta'$ then $g^{-1}(\alpha) = -\overline{\alpha} + \delta'$. Write $N(w_A) = \{-\overline{\gamma}_1 + \delta', \dots - \overline{\gamma}_s + \delta', \overline{\beta}_1 + \delta', \dots, \overline{\beta}_r + \delta'\}$ with $\overline{\beta}_i, \overline{\gamma}_i \in \Delta_f^1$. Hence $\psi_1^*(g^{-1}(N(w_A))) = w_0(\sum \overline{\beta_i} - \sum \overline{\gamma_i}) + s\delta_{\mathfrak{k}}$. Since $A = -\overline{N(w_A)} = \{\overline{\gamma_1}, \dots \overline{\gamma_s}, -\overline{\beta_1}, \dots, -\overline{\beta_r}\}$ we have that $s = |w_0(A) \cap \Delta^+(\mathfrak{p})| = k_A$. The coefficient k_y is computed similarly.

It remains to check that $k_A = 0$ for all $A \in \Sigma$ if and only if α_p is short. Let W_f denote the Weyl group of Δ_f. Clearly $\{w \in W_f \mid w(\Delta_f^+) \supset \psi_1^{*-1}(\Delta_{\mathfrak{k}}^+)\} \subset W'_{\sigma,1}$. By [4, Theorem 5.12], $|W'_{\sigma,1}| = |\{w \in W_f \mid w(\Delta_f^+) \supset \psi_1^{*-1}(\Delta_{\mathfrak{k}}^+)\}|$ if and only if α_p is short. The result follows. □

The non equal rank case with $\mathbf{a_0 = 1}$. It is clear that $\mathfrak{k}_\mu$ is σ-stable, hence we can consider the subalgebra of $\widehat{\mathfrak{k}}_\mu$

$$\widehat{L}(\mathfrak{k}_\mu, \sigma_{|\mathfrak{k}_\mu}) = L(\mathfrak{k}_\mu, \sigma_{|\mathfrak{k}_\mu}) \oplus \mathbb{C}K' \oplus \mathbb{C}d'.$$

Clearly,

$$\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{k}_\mu, \qquad \mathfrak{p}' = \mathfrak{p} \cap \mathfrak{k}_\mu$$

are, respectively, the 1 and -1 eigenspaces of $\sigma_{|\mathfrak{k}_\mu}$ on $\mathfrak{k}_\mu$. Let us denote by $\Delta_{\mathfrak{k}'}$ the $\mathfrak{h}_0$-roots of $\mathfrak{k}'$, by $\Delta(\mathfrak{p}')$ the set of weights of $\mathfrak{h}_0$ on $\mathfrak{p}'$.

Since $\sigma_{|\mathfrak{k}_\mu} = \exp(\pi i ad(\varpi_p))$, it is clear that

$$\Delta_{\mathfrak{k}'} = \Delta_f^0, \qquad \Delta(\mathfrak{p}') \setminus \{0\} = \Delta_f^1. \tag{4.22}$$

For $w \in \widehat{W}_{\mathfrak{k}_\mu}$, let $N_\mu(w) = \{\alpha \in \widehat{\Delta}_\mu^+ \mid w^{-1}(\alpha) < 0\}$. Observe that $W'_{\sigma,1} = \{u \in \widehat{W}_{\mathfrak{k}_\mu} \mid N_\mu(u) \subset \widehat{\Delta}_\mu \setminus \widehat{\Delta}_{\sigma,1}\}$. This is because both $W'_{\sigma,1}$ and $\{u \in \widehat{W}_{\mathfrak{k}_\mu} \mid N_\mu(u) \subset \widehat{\Delta}_\mu \setminus \widehat{\Delta}_{\sigma,1}\}$ are the set of minimal length coset representatives. We notice that the set of real roots in $\widehat{\Delta}_\mu \setminus \widehat{\Delta}_{\sigma,1}$ equals the set of real roots in $\widehat{\Delta}' \setminus \widehat{\Delta}'_{\sigma,1}$. The above observation implies that $N_\mu(u) = N'(u)$ for $u \in W'_{\sigma,1}$. In particular

$$(u\widehat{\rho}' - \widehat{\rho}') = u\widehat{\rho}_\mu - \widehat{\rho}_\mu,$$

where $\widehat{\rho}_\mu$ denotes the sum of the fundamental weights of $\widehat{\mathfrak{k}}_\mu$.

This time the isomorphism

$$t^{\varpi_p} : L(\mathfrak{k}_\mu, id, 2) \to \widehat{L}(\mathfrak{k}_\mu, \sigma_{|\mathfrak{k}_\mu})$$

induces a linear isomorphism $g : \mathfrak{h}_0 \oplus \mathbb{C}\delta' \to \mathfrak{h}_0 \oplus \mathbb{C}\delta'$, still given by (4.18), such that $g(\widehat{\Delta}_\mu)$ is the set $\widehat{\Delta}_{\mu,\sigma}$ of roots of $\widehat{L}(\mathfrak{k}_\mu, \sigma_{|\mathfrak{k}_\mu})$. By Lemma (4.9), we can uniquely extend g to an extended isomorphism of $\widehat{\Delta}_\mu$ with $\widehat{\Delta}_{\mu,\sigma}$, which we still denote by g.

We choose $g\widehat{\Pi}_\mu$ as a set of simple roots for $\widehat{\Delta}_{\mu,\sigma}$, and denote by $\widehat{\Delta}^+_{\mu,\sigma}$ the corresponding positive system of roots. Then it is clear that g maps $\widehat{\Delta}^+_\mu$ onto $\widehat{\Delta}^+_{\mu,\sigma}$. We denote by $\widehat{W}_{\mathfrak{k}_\mu,\sigma}$ the Weyl group of $\widehat{L}(\mathfrak{k}_\mu, \sigma_{|\mathfrak{k}_\mu})$ and, for $w \in \widehat{W}_{\mathfrak{k}_\mu,\sigma}$, we denote by N_σ its negative set with respect to $\widehat{\Delta}^+_{\mu,\sigma}$. By Lemma 4.8, $\widehat{W}_{\mathfrak{k}_\mu,\sigma} = g\widehat{W}_{\mathfrak{k}_\mu} g^{-1}$. Moreover, it is clear that $N_\sigma(gug^{-1}) = gN_\mu(u)$, for all $u \in \widehat{W}_{\mathfrak{k}_\mu}$.

Since $W'_{\sigma,1} = \{u \in \widehat{W}_{\mathfrak{k}_\mu} \mid N_\mu(u) \subset \Delta_{ni} + \mathbb{Z}\delta'\}$, by (4.2), we have that $gW'_{\sigma,1}g^{-1} = \{v \in \widehat{W}_{\mathfrak{k}_\mu,\sigma} \mid N_\sigma(v) \subset \Delta^1_{f,l} + (1+2\mathbb{Z})\delta'\}$. Since the set of real roots in $\widehat{\Delta}_{\mu,\sigma} \setminus (\Delta(\mathfrak{k}') + 2\mathbb{Z}\delta')$ is $\Delta^1_f + (1+2\mathbb{Z})\delta'$ we have in particular that $gW'_{\sigma,1}g^{-1}$ is precisely the set of all elements v in $W'_{\sigma_{|\mathfrak{k}_\mu},0}$ such that $N_\sigma(v) \subset \Delta_{ni} + (1+2\mathbb{Z})\delta'$. We actually have a stronger result.

Lemma 4.11. *If $v \in \widehat{W}_{\mathfrak{k}_\mu,\sigma}$ is such that $N_\sigma(v) \subset \Delta_{ni} + (1+2\mathbb{Z})\delta'$, then v is $\sigma_{|\mathfrak{k}_\mu}$-minuscule. In particular*

$$gW'_{\sigma,1}g^{-1} = \{v \in \mathcal{W}^{\sigma_{|\mathfrak{k}_\mu}}_{ab} \mid \overline{N_\sigma(v)} \subset \Delta_{ni}\}.$$

Proof. We recall (see [4]) that, if $\sigma_{|\mathfrak{k}_\mu}$ is of type $(s_0, \dots, s_n; k)$, then v is $\sigma_{|\mathfrak{k}_\mu}$-minuscule if $ht_{\sigma_{|\mathfrak{k}_\mu}}(\alpha) = 1$ for all $\alpha \in N_\sigma(v)$, where $ht_{\sigma_{|\mathfrak{k}_\mu}}(\alpha) = \sum s_i m_i(\alpha)$.

We use the well known fact that in a finite root system a long root is the sum of two short roots. Suppose now that $N_\sigma(v) \subset \Delta_{ni} + (1+2\mathbb{Z})\delta'$ and that $(2m+1)\delta' + \alpha$ is in $N_\sigma(v)$. By (4.2) $\alpha \in \Delta^1_f$, thus we can write $\alpha = \beta + \gamma$ with $\beta \in \Delta^1_{f,s}$ and $\gamma \in \Delta^0_{f,s}$. It follows that $(2m+1)\delta' + \alpha = (2m\delta' + \gamma) + (\delta' + \beta)$, hence, by the biconvexity property of $N_\sigma(v)$, we find that $\delta' + \beta \in N_\sigma(v)$ unless $m = 0$ and $\gamma \notin \widehat{\Delta}^+_{\mu,\sigma}$. If we write $\alpha = \sum_{i=1}^n m_i \overline{\alpha}_i$, then $m_p = \pm 1$. Since $\delta' + \alpha = \sum_{i=1}^n m_i \alpha_i$ if $m_p = 1$ and $\delta' + \alpha = 2\delta' + \sum_{i=1}^n m_i \alpha_i$ if $m_p = -1$, we see that, in any case, $ht_{\sigma_{|\mathfrak{k}_\mu}}(\delta' + \alpha) = 1$. □

We identify $\Delta_{\mathfrak{k}'}$ with the roots in $\alpha \in \widehat{\Delta}_{\mu,\sigma}$ such that $\alpha(d') = 0$ and choose $\Delta^+_{\mathfrak{k}'} = \widehat{\Delta}^+_{\mu,\sigma} \cap \Delta_{\mathfrak{k}'}$ as a set of positive roots for $\mathfrak{k}'$. We denote by $\mathfrak{b}'$ the corresponding Borel subalgebra of $\mathfrak{k}'$.

Remark 4.4. By the definition of g we see that the set of simple roots for $\mathfrak{k}'$ is given by

$$\Pi_{\mathfrak{k}'} = \begin{cases} \{\overline{\alpha}_i \mid i \neq 0, p\} & \text{if } \overline{\theta}_f(\varpi_p) < 2 \\ \{-\overline{\theta}_f\} \cup \{\overline{\alpha}_i \mid i \neq 0, p\} & \text{if } \overline{\theta}_f(\varpi_p) = 2. \end{cases}$$

It follows that $\mathfrak{b}' = \mathfrak{b}_0 \cap \mathfrak{k}'$.

Combining Lemma 4.11 and Remark 4.4 with the results of [4] exposed in § 3.1, we find the analogue of Theorem 3.9 for this case. Let Σ'_{ni} be the the set of $\mathfrak{b}'$-stable abelian noncompact subspaces of $\mathfrak{p}'$. Recall that in section 2.3 we set $L = N - n$ and $l = \lfloor \frac{N-n}{2} \rfloor$.

Theorem 4.12. *Set* $m = \lfloor \frac{dim \mathfrak{p}}{2} \rfloor$.
(1) Assume that m *is even. Then*

$$L(\tilde{\Lambda}_{m-1}) = L(\tilde{\Lambda}_m) = 2^{l-1} \bigoplus_{A \in \Sigma'_{ni}} L\left(\Lambda_{0,\mathfrak{k}} + \rho_n + w_0\langle A \rangle - k_A \delta_{\mathfrak{k}}\right).$$

(2) Assume that m *is odd. We have*

$$L(\tilde{\Lambda}_m) = 2^l \bigoplus_{A \in \Sigma'_{ni}} L\left(\Lambda_{0,\mathfrak{k}} + \rho_n + w_0\langle A \rangle - k_A \delta_{\mathfrak{k}})\right).$$

In both cases $k_A = |w_0(A) \cap \Delta^+(\mathfrak{p})|$. *Moreover the highest weight vectors of each component indexed by* $A \in \Sigma'_{ni}$ *are, up to a constant factor, the pure spinor (of the spin representation of* $Cl_r(\tilde{\mathfrak{p}})$*)*

$$\prod_{s=0}^{l} \prod_{l+1 \le j_1 < \cdots < j_s \le L} v_{-r'-1, j_k} \prod_{\alpha \in w_0(A) \cap \Delta^+(\mathfrak{p})} (t^{-r'-2} e_\alpha) \prod_{\alpha \in w_0(A) \cap (-\Delta^+(\mathfrak{p}))} (t^{-r'-1} e_\alpha) \tag{4.23}$$

if l *is even, while, if* l *is odd, they are*

$$\prod_{s=0}^{l} \prod_{l+1 < j_1 < \cdots < j_s \le L} v_{-r'-1, j_k} (\prod_{\alpha \in w_0(A) \cap \Delta^+(\mathfrak{p})} \Big((t^{-r'-2} e_\alpha)(v_{-r'-1, l+1})\Big)$$
$$\prod_{\alpha \in w_0(A) \cap (-\Delta^+(\mathfrak{p}))} \Big((t^{-r'-1} e_\alpha)(t^{-r'-1} v_{-r'-1, l+1})\Big).$$

Proof. By a direct computation, we see that $g^{-1}(-\alpha + \delta') = -\alpha + \delta'$ if $\alpha \in \Delta_f^+$, while $g^{-1}(-\alpha + \delta') = -\alpha$ if $\alpha \in -\Delta_f^+$. We can therefore apply the proof of Theorem 4.10. We need only to check the decomposition of $X_r^+ = L(\tilde{\Lambda}_m)$ and $X_r^- = L(\tilde{\Lambda}_{m-1})$ when m is even, but this follows readily from the description of the highest vectors. □

The $A_{2n}^{(2)}$-case. Recall that in this case $L'(\mathfrak{g},\sigma)=\widehat{L}(\mathfrak{g},\sigma)^\vee$, and that we chose $\widehat{\Pi}'=\{\frac{1}{2}(\delta'-\overline{\theta}_f),\overline{\alpha}_1,\dots,\overline{\alpha}_n\}$ as root basis for $L'(\mathfrak{g},\sigma)$. From the explicit description of $\widehat{\Delta}_{\sigma,1}$, we obtain that $W'_{\sigma,1}$ is the set of all elements $v\in\widehat{W}_{\mathfrak{k}_\mu}$ such that $N'(v)$ is included in the set of short roots of $\widehat{\Delta}'$. If we choose $(\widehat{\Pi}')^\vee=\{\delta'-\overline{\theta}_f,\overline{\alpha}_1,\dots,\overline{\alpha}_{n-1},\frac{1}{2}\overline{\alpha}_n\}$ as root basis for $\widehat{L}(\mathfrak{g},\sigma)$, we obtain $(\widehat{\Delta}'^+)^\vee$ as positive system for $\widehat{L}(\mathfrak{g},\sigma)$. Observe that $\widehat{W}=\widehat{W}_{\mathfrak{k}_\mu}$. It is clear that if we regard $v\in\widehat{W}_{\mathfrak{k}_\mu}$ as an element of $\widehat{W}$ and denote by $N^\vee(v)$ the negative set of v with respect to this choice of the positive roots, we obtain that $N^\vee(v)=(N'(v))^\vee$. In particular, for $v\in W'_{\sigma,1}$, $N^\vee(v)=2N'(v)$. Therefore, as a subset on $\widehat{W}$, $W'_{\sigma,1}$ is the set of all elements in v such that $N^\vee(v)$ is included in the set of long roots of $\widehat{\Delta}$. Now we observe that $\Delta_{\mathfrak{k}}=\frac{1}{2}\Delta_{f,l}\cup\Delta_{f,s}=(\Delta_f)^\vee$, so $\{\overline{\alpha}_1,\dots,\overline{\alpha}_{n-1},\frac{1}{2}\overline{\alpha}_n\}$ is the set of simple roots corresponding to $\Delta_{\mathfrak{k}}\cap(\frac{1}{2}\Delta_f^+\cup\Delta_f^+)$. It follows that $w_0(\{\overline{\alpha}_1,\dots,\overline{\alpha}_{n-1},\frac{1}{2}\overline{\alpha}_n\})=\{\alpha_0,\dots,\alpha_{n-1}\}$. Since $\overline{\theta}_f=2\overline{\alpha}_1+\cdots+2\overline{\alpha}_{n-1}+\overline{\alpha}_n$ we see that $w_0(\delta'-\overline{\theta}_f)=\delta'+\overline{\alpha}_n=\alpha_n$, hence $w_0((\widehat{\Pi}')^\vee)=\widehat{\Pi}$ and $w_0((\widehat{\Delta}'^+)^\vee)=\widehat{\Delta}^+$. This says that $w_0W'_{\sigma,1}w_0^{-1}$ is the set of elements of $\widehat{W}$ such that $N(v)$ is included in the set of long roots of $\widehat{\Delta}$. Since the set of long roots of $\widehat{\Delta}$ is $\Delta_{ni}+(1+2\mathbb{Z})\delta'$ we have that $w_0^{-1}W'_{\sigma,1}w_0\subset W'_{\sigma,0}$. Lemma 4.11 applies, so we can conclude that

$$w_0^{-1}W'_{\sigma,1}w_0=\{v\in\mathcal{W}_{ab}^\sigma\mid\overline{N(v)}\subset\Delta_{ni}\}.$$

Arguing as in the previous twisted cases, we finally obtain the analogous of Theorem 3.9 for this case. Set Σ_{ni} to be the set of noncompact $\mathfrak{b}_0$-stable abelian subspaces of $\Delta(\mathfrak{p})$.

Theorem 4.13. *Set $m=\lfloor\frac{dim\mathfrak{p}}{2}\rfloor$.*
(1) Assume that m is even. Then

$$L(\tilde{\Lambda}_{m-1})=L(\tilde{\Lambda}_m)=2^{\frac{n}{2}-1}\bigoplus_{A\in\Sigma_{ni}}L\left(\Lambda_{0,\mathfrak{k}}+\rho_n+\langle A\rangle-|A|\delta_{\mathfrak{k}}\right).$$

(2) Assume that m is odd. We have

$$L(\tilde{\Lambda}_m)=2^{\lfloor\frac{n}{2}\rfloor}\bigoplus_{A\in\Sigma_{ni}}L\left(\Lambda_{0,\mathfrak{k}}+\rho_n+\langle A\rangle-|A|\delta_{\mathfrak{k}})\right).$$

Moreover the highest weight vectors of each component indexed by $A\in\Sigma_{ni}$ are, up to a constant factor, the pure spinor (of the spin representation of $Cl_r(\tilde{\mathfrak{p}})$)

$$(\prod_{s=0}^{\frac{n}{2}}\prod_{\frac{n}{2}+1\le j_1<\cdots<j_s\le n}v_{-r'-1,j_k})\prod_{\alpha\in A}(t^{-r'-2}e_\alpha)$$

if n is even, while, if n is odd, are

$$\prod_{s=0}^{\lfloor \frac{n}{2} \rfloor} \prod_{\frac{n+1}{2}<j_1<\cdots<j_s\leq n} v_{-r'-1,j_k} \prod_{\alpha\in A}(t^{-r'-2}e_\alpha)(t^{-r'-1}v_{-r'-1,l+1}),$$

where $l = \lfloor \frac{n}{2} \rfloor$.

Proof. We know that

$$ch(X_r) = \sum_{u\in W'_{\sigma,1}} ch(L(\Lambda_{0,\mathfrak{k}} + \rho_n - a_0\psi_1^*(\langle N'(u)\rangle))).$$

By the above discussion $a_0\psi_1^*(\langle N'(u)\rangle) = \psi_1^*(\langle N(w_0^{-1}uw_0)\rangle)$ so we can write

$$ch(X_r) = \sum_{A\in\Sigma_{ni}} ch(L(\Lambda_{0,\mathfrak{k}} + \rho_n - \psi_1^*(\langle N(w_A)\rangle))).$$

The coefficient of $\delta_{\mathfrak{k}}$ is computed as in 3.1. The rest of the proof follows as in the previous cases. □

5 The Hermitian symmetric case

In this section we discuss the decomposition of a conformal pair $(so(\mathfrak{p}), \mathfrak{k})$ when $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an infinitesimal Hermitian symmetric space. In this case there exists a node $i \neq 0$ such that $a_i = 1$, $s_0 = s_i = 1$, and $s_j = 0$ for $j \neq 0, i$. It turns out that $\mathfrak{k}$ is an equal rank subalgebra of $\mathfrak{g}$ and it is not semisimple. We can write $\mathfrak{k} = \sum_{S>0} \mathfrak{k}_S \oplus \mathfrak{k}_0$, where $\mathfrak{k}_0 = \mathbb{C}\varpi_i$ and ϖ_i is the unique element of $\mathfrak{h}_0$ such that $\overline{\alpha}_j(\varpi_i) = \delta_{ij}$ for $j > 0$. Recall that in this case

$$\widehat{\mathfrak{k}} = \widehat{[\mathfrak{k}, \mathfrak{k}]} \oplus \widehat{\mathfrak{k}}_0,$$

where $\widehat{\mathfrak{k}}_0 = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{k}_0 \oplus \mathbb{C}K_0$ with bracket defined by

$$[t^n \otimes H + aK_0, t^m \otimes H + bK_0] = \delta_{n,-m}(H, H)_n K_0.$$

As before $(\cdot, \cdot)_n$ is the normalized invariant form of $\mathfrak{g}$. Let $\overline{r} = 0$ if r is even, $\overline{r} = 1$ if r is odd, and $\psi_{\overline{r}}^*$, $\widehat{W}_{\sigma,\overline{r}}$ be defined as in Section 3 or 4, according to the parity of r (note that in this case $a_0 = k = 1$). Then let $\psi_{\overline{r}}^* : \widehat{\mathfrak{h}}^* \to (\widehat{\mathfrak{h}}_{\mathfrak{k}})^*$ denote the transpose of the map $\psi_{\overline{r}}$ restricted to $\widehat{\mathfrak{h}}_{\mathfrak{k}}$.

The same computation performed in the equal rank case for $\mathfrak{k}$ semisimple would give

$$ch(X_r^{\pm}) = \sum_{\substack{u\in W'_{\sigma,\overline{r}} \\ \ell(u)\equiv\epsilon \bmod 2}} ch(L(\psi_{\overline{r}}^*(u\widehat{\rho}_{\overline{r}}) - \widehat{\rho}_{\mathfrak{k}}))), \tag{5.1}$$

where $\epsilon = 0, 1$ according to whether we are considering the $+$ or $-$ case. Moreover $W'_{\sigma,\overline{r}}$ is the set of minimal right coset representatives for $\widehat{W}/\widehat{W}_{\sigma,0}$ if $\overline{r} = 0$ and for $\widehat{W}_{\mathfrak{k}_\mu}/\widehat{W}_{\sigma,1}$ if $\overline{r} = 1$, and $\widehat{\rho}_0 = \widehat{\rho}$, $\widehat{\rho}_1 = \widehat{\rho}'$.

We first deal with the basic and vector case and then we transfer our results to the spin case via map g defined in (4.18). So we assume $\overline{r} = 0$. The starting point to provide a more explicit form of (5.1) is a remarkable subset of stable subpaces which has been introduced in [4], Section 6. Recall from (3.9) the definition of the polytope encoding $\mathfrak{b}_0$-stable abelian subspaces of $\mathfrak{p}$ and set

$$D'_\sigma = D_\sigma \cap \{x \in \mathfrak{h}_1^* \mid (x, \alpha_i) < 0\}.$$

D'_σ corresponds exactly to the set of stable abelian subspaces of $\mathfrak{p}$ which include $\mathfrak{g}_{-\alpha_i}$. Let $\omega_i^\vee$ be the unique element in $Span_{\mathbb{R}}(\alpha_1, \dots, \alpha_n)$ such that $(\alpha_j, \omega_i^\vee) = \delta_{ij}$. From the proof of Lemma 6.1 of [4] we deduce the following fact.

Lemma 5.1. *Consider the group of translations $T_{\mathbb{Z}\omega_i^\vee} = \{t_{j\omega_i^\vee} \mid j \in \mathbb{Z}\}$. Then $\overline{D}'_\sigma$ is a fundamental domain for the action of $T_{\mathbb{Z}\omega_i^\vee}$ on $\bigcup_{w \in W'_{\sigma,0}} w\overline{C}_1$.*

Therefore there exists a "special" subset of stable subspaces of $\mathfrak{p}$ such that the translates of the corresponding alcoves cover the domain $W'_{\sigma,0}\overline{C_1}$. At this point this fact gives little information on the weights appearing in the decomposition (5.1), since $T_{\mathbb{Z}\omega_i^\vee}$ is not included in $\widehat{W}$. This requires some more work, which we perform in a general setting.

Let $\widehat{\mathfrak{l}} = \mathfrak{g}(A)$, where A is a generalized Cartan matrix of affine tipe $X_m^{(1)}$, $\widehat{\mathfrak{h}}_{\mathfrak{l}}$ its Cartan subalgebra, $\widehat{\Pi}_{\mathfrak{l}} = \{\beta_0, \beta_1, \dots \beta_m\}$ and $\widehat{\Pi}_{\mathfrak{l}}^\vee = \{\beta_0^\vee, \beta_1^\vee, \dots \beta_m^\vee\}$ the sets of simple roots and coroots. Moreover, let $\widehat{W}_{\mathfrak{l}}$ be the Weyl group of $\widehat{\mathfrak{l}}$, $\Lambda_0^{\mathfrak{l}}, \Lambda_1^{\mathfrak{l}}, \dots, \Lambda_m^{\mathfrak{l}}$ be the fundamental weights, and $\widehat{\rho}_{\mathfrak{l}} = \Lambda_0^{\mathfrak{l}} + \cdots + \Lambda_m^{\mathfrak{l}}$.

As usual we assume that $\Pi_{\mathfrak{l}} = \{\beta_1, \dots \beta_m\}$ has Dynkin diagram of finite type X_m, and we denote by $\mathfrak{l}$ the corresponding finite dimensional simple Lie subalgebra of $\widehat{\mathfrak{l}}$. Also, we denote by $W_{\mathfrak{l}}$ the Weyl group of $\mathfrak{l}$, by $\Delta_{\mathfrak{l}}^+$ its set of positive roots, by $\theta_{\mathfrak{l}}$ its highest root, and we set $\delta_{\mathfrak{l}} = \beta_0 - \theta_{\mathfrak{l}}$.

Identify $\widehat{\mathfrak{h}}_{\mathfrak{l}}$ and $\widehat{\mathfrak{h}}_{\mathfrak{l}}^*$ via the normalized invariant form. Let $\omega_1^\vee, \dots, \omega_m^\vee$ be the fundamental coweights of $\mathfrak{l}$ and, for $i \in \{1, \dots, m\}$, let $w_i \in W_{\mathfrak{l}}$ be such that $N(w_i) = \{\alpha \in \Delta_{\mathfrak{l}}^+ \mid (\alpha, \omega_i^\vee) \neq 0\}$. It is well-known that w_i exists (and it is unique). We denote by $\widetilde{W}$ the extended affine Weyl group of $\mathfrak{l}$, i.e. $\widetilde{W} = T_{P_{\mathfrak{l}}^\vee} \rtimes W_{\mathfrak{l}}$, where $P_{\mathfrak{l}}^\vee$ is the coweight lattice. We regard $\widetilde{W}$ as a group of transformations on $\widehat{\mathfrak{h}}_{\mathfrak{l}}^*$. Moreover, we set

$$Z = \{t_{\omega_i^\vee} w_i \mid i \in \{1, \dots, m\},\ (\theta_{\mathfrak{l}}, \omega_i^\vee) = 1\} \cup \{1\}.$$

It is well known that Z is exactly the subgroup of all elements in $\widetilde{W}$ that map the fundamental alcove of $\mathfrak{l}$ to itself (see [11]). We may identify the fundamental alcove of $\mathfrak{l}$ with $C_{\widehat{\mathfrak{l}}} \cap \mathfrak{h}_1^*$, where $C_{\widehat{\mathfrak{l}}}$ is the fundamental chamber of $\widehat{\mathfrak{l}}$, and $\mathfrak{h}_1^* = (\Lambda_0^{\mathfrak{l}} + \mathrm{Span}_{\mathbb{R}}\{\beta_0, \dots, \beta_m\})/\mathbb{R}\delta_{\mathfrak{l}}$ (see [12], Section 6.6, or [5], Section 1). Since the restriction to $\mathfrak{h}_1^*$ is a faithful representation of $\widetilde{W}$, we obtain

$$Z = \{v \in \widetilde{W} \mid v\widehat{\Pi}_{\mathfrak{l}} = \widehat{\Pi}_{\mathfrak{l}}\}.$$

Lemma 5.2. *For all* $v \in Z$*,*

$$v\widehat{\rho}_{\mathfrak{l}} = \widehat{\rho}_{\mathfrak{l}}.$$

Proof. We fix $v \in Z \setminus \{1\}$ and set $v^{-1}(\beta_i) = \beta_{j_i}$ for $i \in \{0, 1 \dots, m\}$. We denote by $(\ ,\ )$ the form induced on $\widehat{\mathfrak{h}}_{\mathfrak{l}}^*$ by the normalized invariant form of $\widehat{\mathfrak{l}}$ and we recall that $(\ ,\ )$ is invariant under $\widetilde{W}$. Then, for $i = 0, \dots, m$,

$$v\widehat{\rho}_{\mathfrak{l}}(\beta_i^\vee) = \frac{2(v\widehat{\rho}_{\mathfrak{l}}, \beta_i)}{(\beta_i, \beta_i)} = \frac{2(\widehat{\rho}_{\mathfrak{l}}, v^{-1}\beta_i)}{(v^{-1}\beta_i, v^{-1}\beta_i)} = \widehat{\rho}_{\mathfrak{l}}(\beta_{j_i}^\vee) = 1.$$

It follows that $v\widehat{\rho}_{\mathfrak{l}} \equiv \widehat{\rho}_{\mathfrak{l}} \mod \delta_{\mathfrak{l}}$.

It remains to prove that $(v\widehat{\rho}_{\mathfrak{l}}, \Lambda_0^{\mathfrak{l}}) = 0$. We assume that $v = t_{\omega_i^\vee} w_i$. Since $W_{\mathfrak{l}}$ fixes $\Lambda_0^{\mathfrak{l}}$, by formula (6.5.2) of [12] we have

$$v\Lambda_0^{\mathfrak{l}} = t_{\omega_i^\vee}\Lambda_0^{\mathfrak{l}} = \Lambda_0^{\mathfrak{l}} + \omega_i^\vee - \frac{1}{2}|\omega_i^\vee|^2\delta_{\mathfrak{l}}. \tag{5.2}$$

Since $\widehat{\rho}_{\mathfrak{l}} = \rho_{\mathfrak{l}} + h_{\mathfrak{l}}^\vee \Lambda_0^{\mathfrak{l}}$, where $\rho_{\mathfrak{l}}$ is the sum of fundamental weights of $\mathfrak{l}$ and $h_{\mathfrak{l}}^\vee$ is its dual Coxeter number, we obtain

$$v\widehat{\rho}_{\mathfrak{l}} = v\rho_{\mathfrak{l}} + h_{\mathfrak{l}}^\vee(\Lambda_0^{\mathfrak{l}} + \omega_i^\vee - \frac{1}{2}|\omega_i^\vee|^2\delta_{\mathfrak{l}}). \tag{5.3}$$

But $v\widehat{\rho}_{\mathfrak{l}} - (\rho_{\mathfrak{l}} + h_{\mathfrak{l}}^\vee\Lambda_0^{\mathfrak{l}}) \in \mathbb{R}\delta_{\mathfrak{l}}$, hence

$$v\rho_{\mathfrak{l}} = \rho_{\mathfrak{l}} - h_{\mathfrak{l}}^\vee\omega_i^\vee + x\delta_{\mathfrak{l}}, \tag{5.4}$$

for some $x \in \mathbb{R}$. It follows that

$$w_i\rho_{\mathfrak{l}} = t_{-\omega_i^\vee}(\rho_{\mathfrak{l}} - h_{\mathfrak{l}}^\vee\omega_i^\vee + x\delta_{\mathfrak{l}}) = \rho_{\mathfrak{l}} - h_{\mathfrak{l}}^\vee\omega_i^\vee + x\delta_{\mathfrak{l}} - (\rho_{\mathfrak{l}} - h_{\mathfrak{l}}^\vee\omega_i^\vee + x\delta_{\mathfrak{l}}, \omega_i^\vee)\delta_{\mathfrak{l}},$$

and since the component of $w_i\rho_{\mathfrak{l}}$ in $\mathbb{R}\delta_{\mathfrak{l}}$ is zero, we obtain

$$x = -(\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) + h_{\mathfrak{l}}^\vee|\omega_i^\vee|^2. \tag{5.5}$$

Combining equations (5.3), (5.4), and (5.5), we have

$$(v\widehat{\rho}_{\mathfrak{l}}, \Lambda_0^{\mathfrak{l}}) = -(\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) + \frac{1}{2}h^\vee|\omega_i^\vee|^2. \tag{5.6}$$

Now, since Z is a group, there exists $i' \in \{1\dots,m\}$ such that $v^{-1} = t_{\omega_{i'}^\vee} w_{i'}$. Therefore, as in (5.2), we obtain

$$v^{-1}\Lambda_0^{\mathfrak{l}} = t_{\omega_{i'}^\vee}\Lambda_0^{\mathfrak{l}} = \Lambda_0^{\mathfrak{l}} + \omega_{i'}^\vee - \frac{1}{2}|\omega_{i'}^\vee|^2\delta_{\mathfrak{l}}.$$

Hence

$$(v\widehat{\rho}_{\mathfrak{l}}, \Lambda_0^{\mathfrak{l}}) = (\widehat{\rho}_{\mathfrak{l}}, v^{-1}\Lambda_0^{\mathfrak{l}}) = (\widehat{\rho}_{\mathfrak{l}}, \omega_{i'}^\vee) - \frac{1}{2}h^\vee|\omega_{i'}^\vee|^2. \tag{5.7}$$

Since $T_{P_{\mathfrak{l}}^\vee}$ is normal in $\widetilde{W}_{\mathfrak{l}}$, $v^{-1} = w_i^{-1}t_{-\omega_i^\vee} = t_{-w_i^{-1}\omega_i^\vee}w_i^{-1}$ and since $T_{P_{\mathfrak{l}}^\vee}W$ is a semidirect product,

$$-w_i^{-1}\omega_i^\vee = \omega_{i'}^\vee \tag{5.8}$$

and $w_i^{-1} = w_{i'}$. This implies, in particular, that $|\omega_i^\vee|^2 = |\omega_{i'}^\vee|^2$, and therefore from (5.6) and (5.7) we obtain that

$$-(\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) + \frac{1}{2}h^\vee|\omega_i^\vee|^2 = (\widehat{\rho}_{\mathfrak{l}}, \omega_{i'}^\vee) - \frac{1}{2}h^\vee|\omega_i^\vee|^2.$$

At this point, in order to conclude, it suffices to prove that

$$(\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) = (\widehat{\rho}_{\mathfrak{l}}, \omega_{i'}^\vee). \tag{5.9}$$

By equation (5.8), we have that $(\widehat{\rho}_{\mathfrak{l}}, \omega_{i'}^\vee) = (-w_i\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee)$, hence

$$(\widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) - (\widehat{\rho}_{\mathfrak{l}}, \omega_{i'}^\vee) = (2\widehat{\rho}_{\mathfrak{l}} + w_i\widehat{\rho}_{\mathfrak{l}} - \widehat{\rho}_{\mathfrak{l}}, \omega_i^\vee) = (\langle\Delta_{\mathfrak{l}}^+\rangle - \langle N(w_i)\rangle, \omega_i^\vee).$$

By the definition of w_i, the last term of the above equalities is zero. This proves (5.9) and hence the lemma. $\square$

Denote by Σ' the set of abelian $\mathfrak{b}_0$-stable subspaces of $\mathfrak{p}$ whose corresponding alcoves lie in D'_σ. The previous lemma is the key to read the weight of a factor appearing in (5.1) in terms of the weight of a subspace in Σ'.

Proposition 5.3. *If $A = wC_1$, $w \in W'_{\sigma,0}$, then there exists a unique $k \in \mathbb{Z}$ and a unique $I \in \Sigma'$ such that*

$$\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} =$$
$$\Lambda_{0,\mathfrak{k}} + \langle I\rangle + kh^\vee\varpi_i + (-\frac{1}{2}\dim(I) + k(|I^+| - |I^-|) - \frac{k^2}{4}\dim(\mathfrak{p}))\delta_{\mathfrak{k}}, \tag{5.10}$$

where $I^\pm = I \cap \pm\Delta^+(\mathfrak{p})$.

Proof. By Lemma 5.1 we have $A = t_{k\omega_i^\vee}(A')$ for a unique $k \in \mathbb{Z}$ and a unique alcove $A' \subset D'_\sigma$. Suppose that $A' = w'C_1$, $w' \in \widehat{W}$. Then $wC_1 = t_{k\omega_i^\vee} w'C_1$, and hence there exists a unique $z \in Z$ such that $w = t_{k\omega_i^\vee} wz$. By Lemma 5.2 and formula (6.5.2) of [12] we thus obtain

$$\begin{aligned}\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} &= \psi_0^*(t_{k\omega_i^\vee} w' z(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} = \psi_0^*(t_{k\omega_i^\vee} w'(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} \\ &= \psi_0^*(t_{k\omega_i^\vee}(w'(\widehat{\rho}) - \widehat{\rho})) + \psi_0^*(t_{k\omega_i^\vee}(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} \\ &= \psi_0^*(t_{k\omega_i^\vee}(\langle I\rangle) - \dim(I)\delta') + \psi_0^*(t_{k\omega_i^\vee}(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}}, \qquad (5.11)\end{aligned}$$

where I is the ideal in Σ' corresponding to w'. Note that $\psi_0^*(\delta') = \frac{1}{2}\delta_{\mathfrak{k}}$ and that $\psi_0^*(\omega_i^\vee) = \nu(\varpi_i) + \frac{|\varpi_i|^2}{2}\delta_{\mathfrak{k}}$. Also remark that

$$(\langle I\rangle, \omega_i^\vee) = |I^+| - |I^-|, \qquad (\widehat{\rho}, \omega_i^\vee) = \frac{dim(\mathfrak{p})}{4}. \qquad (5.12)$$

Combining (5.11), (5.12) and formula (5.6) we get (5.10). □

Denote by $c_{I,k}$ the coefficient of $\delta_{\mathfrak{k}}$ in formula (5.10). For $q \in \mathbb{Z}$, denote by $L(\tilde{\Lambda}_\epsilon)_q$ the eigenspace of eigenvalue q for the action of $\mathfrak{k}_0$ on $L(\tilde{\Lambda}_\epsilon)$.

Remark 5.1. $L(\tilde{\Lambda}_\epsilon)_q$ is non zero if and only if $q \equiv \epsilon \, mod \, 2$. In fact, by (2.10), the weights of $L(\tilde{\Lambda}_\epsilon)$ are of the form $\Lambda_{0.\mathfrak{k}} - \sum_{j=1}^s \gamma_j$, $\gamma_j \in \widehat{\Delta}^+(\mathfrak{p})$, $s \equiv \epsilon \, mod \, 2$. Since $\Delta^+(\mathfrak{p}) = \Delta^1_f$ in this case, we have $(\Lambda_{0.\mathfrak{k}} - \sum_{j=1}^s \gamma_j)(\varpi_i) = s$.

From the previous Proposition it follows that

Theorem 5.4.

$$L(\tilde{\Lambda}_\epsilon)_q = \sum_{\substack{I \in \Sigma' \\ |I^+| - |I^-| \equiv q \, mod \, \frac{\dim(\mathfrak{p})}{2}}} L(\Lambda_{0,\mathfrak{k}} + \langle I\rangle + k_I h^\vee \nu(\varpi_i) + (c_{I,k_I} + \frac{\epsilon}{2})\delta_{\mathfrak{k}}),$$

where $k_I = \frac{2(q - |I^+| + |I^-|)}{\dim(\mathfrak{p})}$.

Proof. Consider the sum

$$\sum_{w \in W'_{\sigma,0}} ch(L(\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}})).$$

This sum makes sense because, given a weight μ, there is only a finite number of elements $w \in W'_{\sigma,0}$ such that μ is a weight of $L(\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}})$. Indeed, if μ occurs in $L(\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}})$, then $\mu = \psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}} - \sum_{\alpha \widehat{\Delta}^+_{\mathfrak{k}}} n_\alpha \alpha$, hence $\mu(\varpi_i) = (\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}})(\varpi_i)$. It follows from Proposition 5.3 that

$$\mu(\varpi_i) = \langle I\rangle(\varpi_i) + kh^\vee \nu(\varpi_i)(\varpi_i)$$

and there is only a finite number of $I \in \Sigma'$ and $k \in \mathbb{Z}$ that satisfy this equation.

We can therefore write

$$
\begin{aligned}
\sum_{w \in W'_{\sigma,0}} ch(L(\psi_0^*(w(\widehat{\rho})) - \widehat{\rho}_{\mathfrak{k}})) &= \sum_{w \in W'_{\sigma,0}} \frac{\sum_{u \in \widehat{W}_{\mathfrak{k}}} \epsilon(u) e^{u\psi_0^*(w\widehat{\rho})}}{D_{\mathfrak{k}}} \\
&= \frac{\sum_{w \in W'_{\sigma,0}} \sum_{u \in \widehat{W}_{\mathfrak{k}}} \epsilon(u) e^{u\psi_0^*(w\widehat{\rho})}}{D_{\mathfrak{k}}} \\
&= \frac{D_{\mathfrak{g}}^+}{D_{\mathfrak{k}}} = ch(X_r).
\end{aligned}
$$

Thus we can write

$$
L(\tilde{\Lambda}_\epsilon) = \sum_{k \in \mathbb{Z}} \sum_{\substack{I \in \Sigma' \\ |I| \equiv \epsilon \, mod\, 2}} L(\Lambda_{0,\mathfrak{k}} + \langle I \rangle + k h^\vee \nu(\varpi_i) + (c_{I,k} + \frac{1}{2}\epsilon)\delta_{\mathfrak{k}}). \tag{5.13}
$$

Observe now that

$$
\begin{aligned}
(\Lambda_{0,\mathfrak{k}} + \langle I \rangle + k h^\vee \nu(\varpi_i) + c_{I,k}\delta_{\mathfrak{k}})(\varpi_i) &= |I^+| - |I^-| + k h^\vee |\varpi_i|^2 \\
&= |I^+| - |I^-| + k \frac{\dim(\mathfrak{p})}{2}.
\end{aligned}
$$

The result follows by collecting in (5.13) the terms with eigenvalue q. $\square$

Arguing as in the semisimple equal rank case we obtain, for the spin representations, the following result.

Theorem 5.5. *Set $m = \lfloor \frac{\dim(\mathfrak{p})}{2} \rfloor$. The eigenvalues of ϖ_i on $L(\tilde{\Lambda}_{m-\epsilon})$ are of the form $\frac{\dim(\mathfrak{p})}{4} + q$, $q \in \mathbb{Z}$, $q \equiv \epsilon \, mod\, 2$. The corresponding eigenspaces decompose as*

$$
L(\tilde{\Lambda}_{m-\epsilon})_{\frac{\dim(\mathfrak{p})}{4}+q} = \sum_{\substack{I \in \Sigma' \\ |I^+| - |I^-| \equiv q \, mod\, \frac{\dim(\mathfrak{p})}{2}}} L(\Lambda_{0,\mathfrak{k}} + \langle I \rangle + \rho_n + k_I h^\vee \nu(\varpi_i) + c'_{I,k_I}\delta_{\mathfrak{k}}),
$$

where $k_I = \frac{2(q - |I^+| + |I^-|)}{\dim(\mathfrak{p})}$ and

$$
c'_{I,k_I} = (k-1)|I^+| - k|I^-| + (k^2 - k)\frac{\dim(\mathfrak{p})}{4}.
$$

6 Examples and applications

6.1 Combinatorial interpretation of decompositions in type C

We want to give a combinatorial interpretation of Theorem 3.5 and Theorem 4.10 for the pair $\mathfrak{g} = sp(V_1 \oplus V_2) \supset sp(V_1) \oplus sp(V_2) = \mathfrak{k}$, where V_1, V_2 are complex vector spaces of dimension $2m, 2n$ respectively. It turns out that in this specific case (and indeed only in this) the decomposition formulas afford bijections between level m representations of $\widehat{sp(2n)}$ and level n representations of $\widehat{sp(2m)}$. This result, in the case of the spin representation, appears as Proposition 2 in [13]. In our general setting we are considering the case of a Lie algebra $\mathfrak{g}$ of type C_{n+m} endowed with an involution σ of type $(0, ...0, 1, 0....0; 1)$, where 1 appears in position m.

Let $P_{n,m}$ denote the set of $(m+1)$-weak compositions of n, i.e. ordered $(m+1)$-tuples $(k_0, \dots, k_m)$ of non negative integers such that $\sum_{i=0}^m k_i = n$. Let also $S_{h,k}$ denote the set of h elements subsets of $\{1, \dots, k\}$. The map $(k_0, \dots, k_m) \mapsto \{k_0 + 1, k_0 + k_1 + 2, \dots, k_0 + \cdots + k_{m-1} + m\}$ is a bijection $\zeta_{n,m} : P_{n,m} \to S_{m,m+n}$. If $c : S_{m,m+n} \to S_{n,m+n}$ is the map which associates to an m-element subset of $\{1, \dots, m+n\}$ its complement, the map $\zeta_{m,n}^{-1} \circ c \circ \zeta_{n,m} : P_{n,m} \to P_{m,n}$ is a bijection, which we denote by $(k_0, \dots, k_m) \mapsto (k'_0, \dots, k'_n)$. Set also $k''_i = k'_{n-i}$, $0 \le i \le n$.

Let $\dot\Lambda_0, \dots, \dot\Lambda_m$, $\ddot\Lambda_0, \dots, \ddot\Lambda_n$ be the fundamental weights of the simple ideals of $\widehat{\mathfrak{k}}$, assuming that both components have the Dynkin diagram displayed as in [12], §4, Table $Aff\ I$.

Proposition 6.1. *Let $\mathfrak{g}$, $\mathfrak{k}$ be as above. The following decomposition formulas for the level 1 modules of $\widehat{so(\mathfrak{p})}$ into irreducible $[\widehat{\mathfrak{k}}, \widehat{\mathfrak{k}}]$-modules hold ($\epsilon = 0, 1$):*

$$L(\tilde\Lambda_{l-\epsilon}) = \bigoplus_{\substack{(k_0,\dots,k_m) \in P_{n,m} \\ \sum_{i=0}^m i\,k_i \equiv \epsilon \bmod 2}} L(k_0\dot\Lambda_0 + \ldots + k_m\dot\Lambda_m) \otimes L(k'_0\ddot\Lambda_0 + \ldots + k'_n\ddot\Lambda_n),$$

$$L(\tilde\Lambda_{\epsilon}) = \bigoplus_{\substack{(k_0,\dots,k_m) \in P_{n,m} \\ \sum_{i=0}^m i\,k_i \equiv \epsilon \bmod 2}} L(k_0\dot\Lambda_0 + \ldots + k_m\dot\Lambda_m) \otimes L(k''_0\ddot\Lambda_0 + \ldots + k''_n\ddot\Lambda_n).$$

The key remark to deduce 6.1 from 3.5 and 4.10 is the following combinatorial interpretation of the sets $N(w)$, $w \in W'_{\sigma,\overline{r}}$. Consider the following rectangle $R_{n,m}$ filled with roots (of $\widehat{L}(\mathfrak{g}, \sigma)$) as displayed in the following

figure for $m=2, n=3$:

$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$	$\alpha_1+\alpha_2$
$\alpha_2+\alpha_3+\alpha_4$	$\alpha_2+\alpha_3$	α_2

Then the sets $N(w)$, $w \in W'_{\sigma,\overline{r}}$ can be described as the sets roots lying in the boxes under any lattice path from the South-West corner of the rectangle to the North-East corner. This is readily checked observing that these sets are biconvex (hence are of the form $N(w)$, for some $w \in \widehat{W}$), that they are either void or intersect Π exactly in α_m (hence are of the form $N(w)$, for some $w \in W'_{\sigma,\overline{r}}$), and finally that they are as many as the above lattice paths, hence $\binom{n+m}{n} = |W'_{\sigma,\overline{r}}|$ in number (see [4], Table 5.1). Now the proposition follows by direct computation taking into account that $\Lambda_{0,\mathfrak{k}} = n\dot{\Lambda}_0 + m\ddot{\Lambda}_0$, $\rho_n + \Lambda_{0,\mathfrak{k}} = n\dot{\Lambda}_m + m\ddot{\Lambda}_0$, $\psi^*_{\overline{r}}(\alpha_m) = \dot{\Lambda}_1 + \ddot{\Lambda}_1$ (r even) and $w_0 = s_m s_{m-1} s_m \cdots s_1 s_2 \cdots s_m$, $\psi^*_{\overline{r}}(\alpha_m) = w_0(\dot{\Lambda}_1) + \ddot{\Lambda}_1$ (r odd) . More explicitely, it is not difficult to see that if p_w is the lattice path associated to $w \in W'_{\sigma,\overline{r}}$ and $p_w \leftrightarrow (a_1,\ldots,a_m) \leftrightarrow (b_1,\ldots,b_n)$, where $0 \le a_1 \le a_2 \le \ldots \le n$ (resp. $m \ge b_1 \ge b_2 \ge \ldots \ge 0$) are the lengths of the rows and (resp. columns) of the subdiagram of $R_{n,m}$ whose bottom border is p_w, counted from bottom to top (resp. from left to right), then

$$\Lambda_{0,\mathfrak{k}} - \langle \psi^*_{\overline{r}}(N(w)) \rangle = \sum_{i=1}^{n} \dot{\Lambda}_{m-b_i} + \sum_{i=1}^{m} \ddot{\Lambda}_{n-a_i}$$

for r even and

$$\rho_n + \Lambda_{0,\mathfrak{k}} - \langle \psi^*_{\overline{r}}(N(w)) \rangle = \sum_{i=1}^{n} \dot{\Lambda}_{b_i} + \sum_{i=1}^{m} \ddot{\Lambda}_{n-a_i}.$$

for r odd.

6.2 A special case

Suppose that σ is an automorphism of type $(0,\ldots,1,\ldots,0;1)$ with 1 in a position corresponding to a long simple root (say α_p). We show below how

to calculate the $\widehat{\mathfrak{k}}$-decomposition of the basic and vector representations in terms of a special class of representatives and how to get information on asymptotic dimension. We set for shortness $\widehat{W} = \widehat{W}_{\sigma,0}$, $W' = W'_{\sigma,0}$. Let W_f the Weyl group generated by $s_1, \dots, s_n$. Let $a(\Lambda)$ denote the asymptotic dimension of a module $L(\Lambda)$ (see [16, (2.1.5)] for the definition).

Proposition 6.2. *1. The map $w \mapsto w_\sigma w$ is an involution i on W'. Moreover we have that $i(W' \cap W_f) = W' \setminus (W' \cap W_f)$.*
2. Denote by $\Lambda_w = \sum_{i=0}^n b_i \Lambda_i$ the weight of the factor indexed by $w \in W'$ in formula (3.8). If $w \in W' \cap W_f$, then $\Lambda_{i(w)} = \sum_{i=0}^n b_i \Lambda_{\pi(i)}$, where π is a suitable permutation of $\{1, \dots, n\}$. In particular $a(\Lambda_w) = a(\Lambda_{i(w)})$.

Proof. Consider the set P_σ defined in (3.10). By [4, Lemma 5.9], , we have $w_\sigma(P_\sigma) \subseteq P_\sigma$, hence left multiplication by w_σ gives a map $i : W' \to W'$. By [4, Lemma 5.11], we deduce that 0 does not belong to $w_\sigma \overline{C}_1$; in particular $w_\sigma \notin W' \cap W_f$. This easily implies that $i(W' \cap W_f) \subseteq W' \setminus (W' \cap W_f)$. It is clear that i is injective. Proposition 5.8 and Theorem 5.12 of [4] give $|W'| = 2|W' \cap W_f|$, hence $i(W' \cap W_f) = W' \setminus (W' \cap W_f)$. Finally i is an involution since w_σ is an involution. Indeed w_σ is defined in [4] as the product of certain elements of the extended Weyl groups of the irreducible components of the extended Dynkin diagram of $\mathfrak{g}$ minus the pth node; in [11] the action of these elements is completely worked out. This explicit description proves both that w_σ is an involution and that it acts on each simple component $\widehat{\mathfrak{k}}_S$ by permuting the fundamental weights. The assertion on asymptotic dimension follows from the fact that this quantity is invariant under the action of certain elements in the extended affine Weyl group. More precisely the invariance follows from [16, (2.2.15-16)] taking into account that w_σ is a product of elements in W_0^+ (in the notation of [16]). □

6.3 More examples

The following examples should make clear how to use our decomposition formulas in explicit cases. To avoid cumbersome notation we describe the decomposition as $[\widehat{\mathfrak{k}}, \widehat{\mathfrak{k}}]$ modules. In other words we consider the weights of the $\widehat{\mathfrak{k}}$-modules appearing in the decompositions modulo $\delta_{\mathfrak{k}}$.

1. We describe the decomposition of X_{-1} when $\mathfrak{g}$ is of type G_2 and σ of type $(0,1,0;1)$. In this case $\widehat{\mathfrak{k}}$ is of type $A_1^{(1)} \times A_1^{(1)}$. $W_{\sigma,1}$ is generated inside $\widehat{W}$ by $s_0, s_2, s_1s_2s_1s_2s_1, s_1s_2s_1s_0s_1s_2s_1s_0s_1s_2s_1$ and

$$W'_{\sigma,1} = \{id, s_1, s_1s_0, s_1s_2, s_1s_2s_0, s_1s_2s_0s_1\}.$$

According to formula 4.10, the highest weights of the irreducible components are of the form $2\dot\Lambda_0 + 10\ddot\Lambda_0 + \rho_n - \psi_1^*(\langle N(u)\rangle)$, where u ranges over $W'_{\sigma,-1}$. Here and in the following $\dot\Lambda_i$ denotes the i-th fundamental weight for the first copy of $A_1^{(1)}$ whereas $\ddot\Lambda_i$ denotes the i-th fundamental weight for the other copy. Since $\rho_n = w_0(2\alpha_1 + 3\alpha_2)$ and $\overline{\alpha}_1 = -\frac{1}{2}(\overline{\alpha}_0 + 3\overline{\alpha}_2)$ we have

$$
\begin{aligned}
X_{-1} =& L(2\dot\Lambda_1) \otimes L(10\ddot\Lambda_0) \oplus \\
& L(\dot\Lambda_0 + \dot\Lambda_1) \otimes L(7\ddot\Lambda_0 + 3\ddot\Lambda_1) \oplus \\
& L(2\dot\Lambda_1) \otimes L(4\ddot\Lambda_0 + 6\ddot\Lambda_1) \oplus \\
& L(2\dot\Lambda_0) \otimes L(6\ddot\Lambda_0 + 4\ddot\Lambda_1) \oplus \\
& L(\dot\Lambda_0 + \dot\Lambda_1) \otimes L(3\ddot\Lambda_0 + 7\ddot\Lambda_1) \oplus \\
& L(2\dot\Lambda_0) \otimes L(10\ddot\Lambda_1).
\end{aligned}
$$

2. We describe the decomposition of X_0 when $\mathfrak{g}$ is of type D_4 and σ of type $(0,1,0,0;2)$. In this case $\widehat{\mathfrak{k}}$ is of type $A_1^{(1)} \times C_2^{(1)}$. $W_{\sigma,0}$ is generated inside $\widehat{W}$ by $s_0, s_2, s_3, s_1s_0s_1s_2s_1s_0s_1$, $s_1s_2s_3s_2s_1s_0s_1s_2s_3s_2s_1$ and we have

$$
\begin{aligned}
W'_{\sigma,0} = \{&Id, s_1, s_1s_0, s_1s_2, s_1s_0s_1, s_1s_0s_2, s_1s_2s_3, s_1s_0s_2s_3, s_1s_2s_3s_2, \\
& s_1s_0s_2s_3s_2, s_1s_2s_3s_2s_1, s_1s_0s_2s_3s_2s_1\}.
\end{aligned}
$$

According to formula 4.12, the highest weights of the irreducible components are of the form $10\dot\Lambda_0 + 3\ddot\Lambda_0 - \psi_0^*(\langle N(u)\rangle)$, where u ranges over $W'_{\sigma,0}$. Taking into account that $\overline{\alpha}_1 = -(\overline{\alpha}_0 + \overline{\alpha}_2 + \overline{\alpha}_3)$,we get

$$
\begin{aligned}
X_0 =& L(10\dot\Lambda_0) \otimes L(3\ddot\Lambda_0) \oplus \\
& L(8\dot\Lambda_0 + 2\dot\Lambda_1) \otimes L(2\ddot\Lambda_0 + \ddot\Lambda_2) \oplus \\
& L(6\dot\Lambda_0 + 4\dot\Lambda_1) \otimes L(\ddot\Lambda_0 + 2\ddot\Lambda_1) \oplus \\
& L(4\dot\Lambda_0 + 6\dot\Lambda_1) \otimes L(\ddot\Lambda_0 + 2\ddot\Lambda_1) \oplus \\
& L(2\dot\Lambda_0 + 8\dot\Lambda_1) \otimes L(2\ddot\Lambda_0 + \ddot\Lambda_2) \oplus \\
& L(10\dot\Lambda_1) \otimes L(3\ddot\Lambda_0) \oplus \\
& L(10\dot\Lambda_0) \otimes L(3\ddot\Lambda_2) \oplus \\
& L(8\dot\Lambda_0 + 2\dot\Lambda_1) \otimes L(\ddot\Lambda_0 + 2\ddot\Lambda_2) \oplus \\
& L(6\dot\Lambda_0 + 4\dot\Lambda_1) \otimes L(2\ddot\Lambda_1 + \ddot\Lambda_2) \oplus \\
& L(4\dot\Lambda_0 + 6\dot\Lambda_1) \otimes L(2\ddot\Lambda_1 + \ddot\Lambda_2) \oplus \\
& L(2\dot\Lambda_0 + 8\dot\Lambda_1) \otimes L(\ddot\Lambda_0 + 2\ddot\Lambda_2) \oplus \\
& L(10\dot\Lambda_1) \otimes L(3\ddot\Lambda_2).
\end{aligned}
$$

3. It is easy to see from our formulas that if $\mathfrak{g}$ is of type D_{l+1} and σ is of type$(1, 0, \dots, 0; 2)$ then both the spin and the basic and vector representations restrict to the spin and basic and vector representations for $B_l^{(1)}$.

4. Finally we consider the decomposition of the spin representation X_{-1} for $\mathfrak{g}$ of type D_4 and σ of type $(0, 1, 0, 0; 2)$. As in example 2, $\widehat{\mathfrak{k}}$ is of type $A_1^{(1)} \times C_2^{(1)}$. $\widehat{W}_{\mathfrak{k}_\mu}$ is an affine Weyl group of type B_3. Recall that we chose $\widehat{\Pi}_\mu = \{-\theta_f + \delta', \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3\}$ as a set of positive roots for $\widehat{\Delta}_\mu$. Set $\beta_0 = -\theta_f + \delta'$, $\beta_i = \overline{\alpha}_i, i = 1, 2, 3$, $s_i = s_{\beta_i}$, $i = 0, 1, 2, 3$. Then

$$\psi_1^{*-1}(\widehat{\Pi}_{\mathfrak{k}}) = \{\beta_2, \beta_3, \beta_0 + \beta_2 + \beta_3, \beta_0 + \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3\},$$

hence $\widehat{W}_{\sigma,1}$ is generated by $s_2, s_3, s_0s_2s_3s_2s_0, s_0s_1s_2s_1s_0, s_1s_2s_3s_2s_1$. A set of minimal right coset representatives is

$$W'_{\sigma,1} = \{Id, s_0, s_1, s_1s_0, s_1s_2, s_0s_2\}.$$

Taking into account that $\overline{\alpha}_1 = -(\overline{\alpha}_0 + \overline{\alpha}_2 + \overline{\alpha}_3)$, and that $\rho_n = 5\dot{\Lambda}_0 + \ddot{\Lambda}_1$, we get

$$\begin{aligned} X_{-1} = & L(5\dot{\Lambda}_0 + 5\dot{\Lambda}_1) \otimes L(2\ddot{\Lambda}_0 + \ddot{\Lambda}_1) \oplus \\ & L(3\dot{\Lambda}_0 + 7\dot{\Lambda}_1) \otimes L(\ddot{\Lambda}_0 + \ddot{\Lambda}_1 + \ddot{\Lambda}_2) \oplus \\ & L(7\dot{\Lambda}_0 + 3\dot{\Lambda}_1) \otimes L(\ddot{\Lambda}_0 + \ddot{\Lambda}_1 + \ddot{\Lambda}_2) \oplus \\ & L(5\dot{\Lambda}_0 + 5\dot{\Lambda}_1) \otimes L(\ddot{\Lambda}_1 + 2\ddot{\Lambda}_2) \oplus \\ & L(\dot{\Lambda}_0 + 9\dot{\Lambda}_1) \otimes L(3\ddot{\Lambda}_1) \oplus \\ & L(9\dot{\Lambda}_0 + \dot{\Lambda}_1) \otimes L(3\ddot{\Lambda}_1). \end{aligned}$$

6.4 Connections with modular invariance.

We now try to use the formulas developed in the previous sections to obtain information on the action of $SL(2, \mathbb{Z})$ on modified characters described in [16]. Here we consider the very special case when σ comes from an automorphism of the diagram of $\mathfrak{g}$. This implies that $\mathfrak{g}$ is either simple of type A, D, E or of complex type. Furthermore $\mathfrak{k}$ is simple. We shall also assume that $\mathfrak{g}$ is not of type A_{2n}. These are precisely the cases in which $W'_{\sigma,1} = \{1\}$.

Let $h_{\mathfrak{k}}^\vee$ denote the dual Coxeter number of $\mathfrak{k}$ and set $j = h^\vee - h_{\mathfrak{k}}^\vee$. We denote by $\dot{\Lambda}_i$ the i-th fundamental weight of $\widehat{\mathfrak{k}}$ and by P_+^j the set of dominant weights for $\widehat{\mathfrak{k}}$ of level j. Recall that $N = rk\, \mathfrak{g}$ while $n = rk\, \mathfrak{k}$. By (3.7) we

have that

$$ch(L(\tilde{\Lambda}_0)) - ch(L(\tilde{\Lambda}_1)) = \sum_{w \in W'_{\sigma,0}} \epsilon(w) ch(L(\psi_0^*(w\widehat{\rho}) - \widehat{\rho}_{\mathfrak{k}})). \tag{6.1}$$

Formula (4.17) becomes in our case

$$ch(L(\tilde{\Lambda}_m)) = 2^{\lfloor \frac{N-n}{2} \rfloor} ch(L(j\dot{\Lambda}_0 + \rho_n)) \tag{6.2}$$

if $N-n$ is odd, and

$$ch(L(\tilde{\Lambda}_{m-1})) + ch(L(\tilde{\Lambda}_m)) = 2^{\lfloor \frac{N-n}{2} \rfloor} ch(L(j\dot{\Lambda}_0 + \rho_n)), \tag{6.3}$$

if $N-n$ is even. Here $m = \lfloor \frac{\dim(\mathfrak{p})}{2} \rfloor$.

Denote by χ_Λ is the modified character of $L(\Lambda)$ (see [16, (1.5.11)]), and set $Y = \{h \in \widehat{\mathfrak{h}}^* \mid Re\, \delta_{\mathfrak{k}}(h) > 0\}$. Moreover we write Λ_w for $\psi_0^*(w\widehat{\rho}) - \widehat{\rho}_{\mathfrak{k}}$. Since the pair $(so(\mathfrak{p}), \mathfrak{k})$ is conformal, relation (6.1) translates into

$$(\chi_{\tilde{\Lambda}_0} - \chi_{\tilde{\Lambda}_1})_{|Y} = \sum_{w \in W'_{\sigma,0}} \epsilon(w) \chi_{\Lambda_w}, \tag{6.4}$$

whereas (6.2) gives

$$(\chi_{\tilde{\Lambda}_m})_{|Y} = 2^{\lfloor \frac{N-n}{2} \rfloor} \chi_{j\dot{\Lambda}_0 + \rho_n} \tag{6.5}$$

($N-n$ odd), and (6.3) gives

$$(\chi_{\tilde{\Lambda}_{m-1}} + \chi_{\tilde{\Lambda}_m})_{|Y} = 2^{\lfloor \frac{N-n}{2} \rfloor} \chi_{j\dot{\Lambda}_0 + \rho_n} \tag{6.6}$$

($N-n$ even). Recall from [16, Remark 4.2.2] that if $N-n$ is odd,

$$\chi_{\tilde{\Lambda}_m}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}} (\chi_{\tilde{\Lambda}_0} - \chi_{\tilde{\Lambda}_1})(\tau) \tag{6.7}$$

and, if $N-n$ is even,

$$(\chi_{\tilde{\Lambda}_{m-1}} + \chi_{\tilde{\Lambda}_m})(-\frac{1}{\tau}) = (\chi_{\tilde{\Lambda}_0} - \chi_{\tilde{\Lambda}_1})(\tau). \tag{6.8}$$

By modular invariance of modified characters,

$$\chi_{j\dot{\Lambda}_0 + \rho_n}(-\frac{1}{\tau}) = \sum_{\Lambda \in P_+^j} a(\Lambda, j\dot{\Lambda}_0 + \rho_n) \chi_\Lambda. \tag{6.9}$$

(here $a(\cdot,\cdot)$ is the function $P_+^j \times P_+^j \to \mathbb{C}$ defined in [16, (2.1.7)]). Assume $N-n$ even and use (6.4),(6.8),(6.6),(6.9) obtaining

$$\sum_{w\in W'_{\sigma,0}} \epsilon(w)\chi_{\Lambda_w}(\tau) = (\chi_{\tilde\Lambda_0} - \chi_{\tilde\Lambda_1})(\tau) = (\chi_{\tilde\Lambda_{m-1}} + \chi_{\tilde\Lambda_m})(-\frac{1}{\tau})$$
$$= 2^{\frac{N-n}{2}}\chi_{j\dot\Lambda_0+\rho_n}(-\frac{1}{\tau}) = 2^{\frac{N-n}{2}} \sum_{\Lambda\in P_+^j} a(\Lambda, j\dot\Lambda_0+\rho_n)\chi_\Lambda(\tau).$$

The case $N-n$ odd is analogous. We can deduce the following

Proposition 6.3. *We have $a(\Lambda, j\dot\Lambda_0+\rho_n) = 0$ unless there exists $w \in W'_{\sigma,0}$ such that $\Lambda + \widehat\rho_{\mathfrak k} = \psi_0^*(w\widehat\rho)$. In such a case $a(\Lambda, j\dot\Lambda_0+\rho_n) = 2^{-\frac{N-n}{2}}(-1)^{\ell(w)}$.*

Remark 6.1. In the complex case this result was obtained in the same way in [16], (4.2.14).

Remark 6.2. Recall that, if Σ is the set of $\mathfrak b_0$-stable abelian subspaces of $\mathfrak p$, then, according to Theorem 3.9, the set Σ parametrizes the irreducible components of X_0. By [16, (2.2.3)], we know that

$$\sum_{\Lambda\in P_+^j} |a(\Lambda, j\dot\Lambda_0+\rho_n)|^2 = 1.$$

We can therefore deduce that $|\Sigma| = 2^{N-n}$ in these cases. This fact was first proved in [22] by a different method. In the complex case we have yet another proof of Peterson's 2^{rank} abelian ideals Theorem (see again [16]).

Remark 6.3. If $\mathfrak g$ is of type D_N, then $\mathfrak k$ is of type B_{N-1} and one only obtains again that $a(\dot\Lambda_{N-1}, \dot\Lambda_0) = -a(\dot\Lambda_{N-1}, \dot\Lambda_1) = \frac{1}{\sqrt 2}$ and $a(\dot\Lambda_{N-1}, \dot\Lambda_{N-1}) = 0$.

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